A Script of the Lecture

Theoretical Physics I
(Mechanics)

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Comments and hints (e.g. errors) please via email to us: mail@skriptweb.de – Thanks.

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1. Mathematical background

Given 2 Vectors $\vec{A}$, $\vec{B}$:

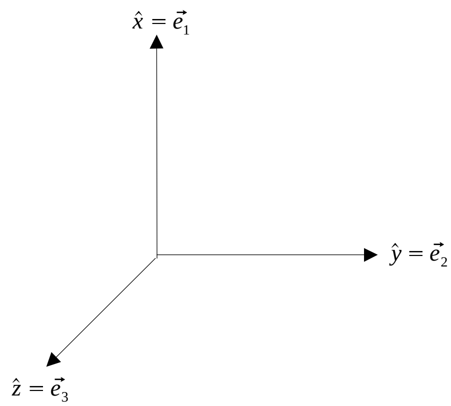
$$\vec{A} \cdot \vec{B} = A \cdot B \cos \theta$$
θ = angle between $\vec{A}$, $\vec{B}$
$$\vec{A} \times \vec{B} = \vec{C} \perp \vec{A} , \vec{B}$$
$$|\vec{C}| = A \; B \sin \theta$$

Vector: components in coordinates

reference frame: cartesian frame

$$\vec{x} = (x_1 , x_2 , x_3) = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3$$

$\vec{e}_i$ are unit vectors along cartesian axes



Summation convention:

$$\vec{x} = x_i \vec{e}_i \triangleq \sum_{i=1}^{3} x_i \vec{e}_i$$

Metric of the space:

$$d = \sqrt{\sum_i (x_i - y_i)^2}$$ **Euclidean metric**

$$\vec{A} = (a_i) , \vec{B} = (b_i)$$
$$\vec{A} \cdot \vec{B} = a_i \; b_i$$
$\vec{A} \times \vec{B} = 0 \quad \Leftrightarrow \quad \vec{A} \uparrow\uparrow \vec{B}$ or $\vec{A} \uparrow\downarrow \vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{e}_1 (a_2 \; b_3 - a_3 \; b_1) - \vec{e}_2 (a_1 \; b_3 - a_3 \; b_1) + \vec{e}_3 (a_1 \; b_2 - a_2 \; b_1)$$
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

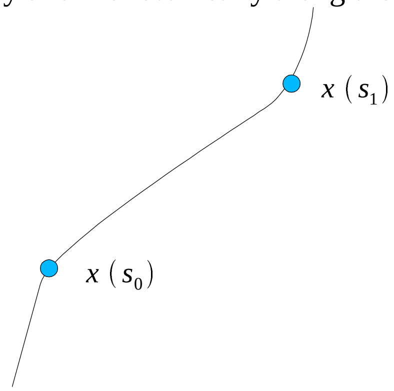
Position vector $\vec{x} = \vec{x}(t)$; $\vec{x}(t) = x_i(t) \vec{e}_i$

$$\frac{\mathrm{d} (\vec{x}(t) \cdot y(t))}{\mathrm{d} t} = \frac{\mathrm{d} \vec{x}}{\mathrm{d} t} \cdot \vec{y} + \vec{x} \cdot \frac{\mathrm{d} \vec{y}}{\mathrm{d} t} = \dot{\vec{x}} \cdot \vec{y} + \vec{x} \cdot \dot{\vec{y}}$$
$$\int \vec{x}(t) \, \mathrm{d} t = \vec{e}_i \int x_i(t) \, \mathrm{d} t$$

1.0 Trajectories

$\vec{x}(t)\,,\ \vec{v}(t) = \dot{\vec{x}}(t)$

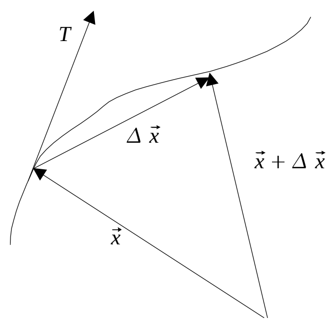
s parameter that increases smoothly and monotonically along the trajectory



$\vec{x}(s_0)$ and $\vec{x}(s_1)$

Distance along the trajectory:

$$l(s_0, s_1) = \int_{s_0}^{s_1} |\vec{x}(s)|\, \mathrm{d}\, s = \int_{s_0}^{s_1} \sqrt{\sum_i x_i^2}\, \mathrm{d}\, s$$



l as parameter

$$\vec{v} = \frac{\mathrm{d}\,\vec{x}}{\mathrm{d}\,l}\frac{\mathrm{d}\,l}{\mathrm{d}\,t}$$

$\vec{T}$ = tangent vector
$\Delta\vec{x}$ = coord vector

$$\vec{\tau} = \frac{\mathrm{d}\,\vec{x}}{\mathrm{d}\,l} = \lim_{l\to 0}\frac{\Delta\,\vec{x}}{\Delta\,l}$$

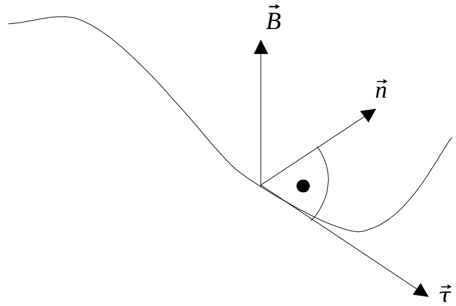
$|\Delta\,\vec{x}| = \Delta\,l \Rightarrow \vec{\tau}$ = unit vector tangent to the trajectory

$\vec{v} = \vec{\tau}\,\frac{\mathrm{d}\,l}{\mathrm{d}\,t} = \vec{\tau}\cdot\dot{l}$ and $\vec{\tau} = \frac{\mathrm{d}\,\vec{x}}{|\mathrm{d}\,\vec{x}|} = \frac{\vec{v}}{|\vec{v}|}$

$\vec{v}$ is everywhere tangent to the trajectory and equal in magnitude to the speed $\dot{l}$ along the trajectory.

Moving reference frames:

Osculating plane (Schmiegungsebene)

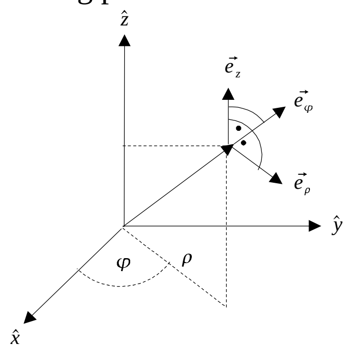


$\vec{n} \perp \vec{\tau}\ ,\ \vec{n} = \frac{\dot{\vec{\tau}}}{|\dot{\vec{\tau}}|}$

(**principal normal vector –** Hauptnormalenvektor)

$\vec{\tau} \cdot \vec{\tau} = 1\ ,\ \frac{d}{d\,t}(\vec{\tau} \cdot \vec{\tau}) = 0 = 2\,\vec{\tau}\,\frac{d\,\vec{\tau}}{d\,t}$

The unit vector $\vec{B}$ normal to the osculating plane is called **binormal vector**. (Binormalenvektor)



1.1 Coordinate systems

Cylidrical polar coordinates

ϱ	radial distance from z
ϕ	angular rotation from x-axis
z	elevation above x-y-plane

Transformations

$\vec{x} = (x\ ,\ y\ ,\ z) = (\varrho \cos\varphi\ ,\ \varrho \sin\varphi\ ,\ z)$

$\varrho = \sqrt{x^2 + y^2}$

$\varphi = \arctan\frac{y}{x}$

Basis vectors: Point in the direction of increasing ρ , φ and z, respectively:

$\vec{e}_\varrho = \frac{\partial\,\vec{x}/\partial\,\varrho}{|\partial\,\vec{x}/\partial\,\varrho|} = \frac{(\cos\varphi\ ,\ \sin\varphi\ ,\ 0)}{1}$

$\vec{e}_\varphi = \frac{\partial\,\vec{x}/\partial\,\varphi}{|\partial\,\vec{x}/\partial\,\varphi|} = \frac{(-\varrho\sin\varphi\ ,\ \varrho\cos\varphi\ ,\ 0)}{\varrho} = (-\sin\varphi\ ,\ \cos\varphi\ ,\ 0)$

$\vec{e}_z = \frac{(\partial\,\vec{x}/\partial\,z)}{|\partial\,\vec{x}/\partial\,z|} = \frac{(0\ ,\ 0\ ,\ 1)}{1}$

$\Rightarrow\ \vec{x} = \varrho\,\vec{e}_\varrho + z\,\vec{e}_z$

Velocity:

$$\dot{\vec{x}} = \dot{\varrho}\ \vec{e}_\varrho + \varrho\ \dot{\vec{e}}_\varrho + \dot{z}\ \vec{e}_z + \underbrace{(z\ \dot{\vec{e}}_z)}_{=0}$$

$$\frac{\mathrm{d}\ \vec{e}_\varrho}{\mathrm{d}\ t} = \frac{\partial\ \vec{e}_\varrho}{\partial\ \varrho}\frac{\mathrm{d}\ \varrho}{\mathrm{d}\ t} + \frac{\partial\ e_\varrho}{\partial\ \varphi}\frac{\mathrm{d}\ \varphi}{\mathrm{d}\ t} + \frac{\partial\ \vec{e}_\varrho}{\partial\ z}\frac{\mathrm{d}\ z}{\mathrm{d}\ t} = 0$$

$$\vec{e}_\varrho = (\cos\ \varphi\ ,\ \sin\ \varphi\ ,\ 0)$$

$$\frac{\partial\ \vec{e}_\varrho}{\partial\ \varphi} = (-\sin\ \varphi\ ,\ \cos\ \varphi\ ,\ 0)$$

$$\frac{\mathrm{d}\ \vec{e}_\rho}{\mathrm{d}\ t} = (-\sin\ \varphi\ ,\ \cos\ \varphi\ ,\ 0)\cdot\dot{\varphi} = \dot{\varphi}\ \vec{e}_\varphi$$

$$\frac{\mathrm{d}\ \vec{e}_\varphi}{\mathrm{d}\ t} = -\dot{\varphi}\ \vec{e}_\varrho$$

$$\frac{\mathrm{d}\ \vec{e}_z}{\mathrm{d}\ t} = 0$$

$$\Rightarrow\ \dot{\vec{x}} = \dot{\varrho}\ \vec{e}_\varrho + \varrho\ \dot{\varphi}\ \vec{e}_\varphi + \dot{z}\ \vec{e}_z$$

$$\ddot{\vec{x}} = (\ddot{\varrho} - \varrho\ \dot{\varphi}^2)\ \vec{e}_\varrho + (\varrho\ \ddot{\varphi} + 2\ \dot{\varrho}\ \dot{\varphi})\ \vec{e}_\varphi + \ddot{z}\ \vec{e}_z$$

1.2 Vector calculus

We define the **del operator**

$$\vec{\nabla} = \frac{\partial}{\partial\ x_1}\ \vec{e}_1 + \frac{\partial}{\partial\ x_2}\ \vec{e}_2 + \frac{\partial}{\partial\ x_3}\ \vec{e}_3$$

Scalar field (such as temperature)

$$\Phi\ (x_1\ ,\ x_2\ ,\ x_3)$$

The **gradient** of Φ is a vector field

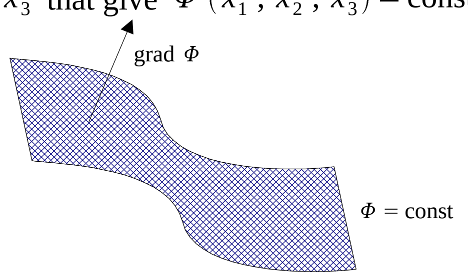
$$\vec{\nabla}\ \Phi = \frac{\partial\ \Phi}{\partial\ x_i}\ \vec{e}_i$$

$\vec{\nabla}\ \Phi$ lies perpendicular to surfaces of constant value $\Phi\ (x_{1,}\ x_{2,}\ x_3) = \mathrm{const}$ which are termed **equipotential surfaces**.

For an arbitrary change in position, $\mathrm{d}\ \vec{x} = (\mathrm{d}\ x_1\ ,\ \mathrm{d}\ x_2\ ,\ \mathrm{d}\ x_3)$, the total derivative of Φ is given by

$$\mathrm{d}\ \Phi = \vec{\nabla}\ \Phi\cdot\mathrm{d}\ \vec{x} = \frac{\partial\ \Phi}{\partial\ x_i}\ \mathrm{d}\ x_i = \left(\frac{\partial\ \Phi}{\partial\ x_1}\ ,\ \frac{\partial\ \Phi}{\partial\ x_2}\ ,\ \frac{\partial\ \Phi}{\partial\ x_3}\right)\cdot(\mathrm{d}\ x_1\ ,\ \mathrm{d}\ x_2\ ,\ \mathrm{d}\ x_3)\ .$$

We consider values of $x_1\ ,\ x_2\ ,\ x_3$ that give $\Phi\ (x_1\ ,\ x_2\ ,\ x_3) = \mathrm{const}$.



$\mathrm{d}\ \Phi = 0 = \vec{\nabla}\ \Phi\cdot\mathrm{d}\ \vec{x}$, where $\mathrm{d}\ \vec{x}$ lies within the equipotential surface. $\Rightarrow\ \Rightarrow\ \vec{\nabla}\ \Phi\perp\mathrm{d}\ \vec{x}$.

$$\vec{\nabla}\ \Phi \equiv \mathrm{grad}\ \Phi$$

Calculate the gradient in terms of cylindrical polar coordinates:

$$\varrho=\sqrt{x^2+y^2}$$

$$\vec{e}_\varrho=\cos\varphi\ \vec{e}_x+\sin\varphi\ \vec{e}_y$$

$$\varphi=\arctan\frac{y}{x}$$

$$\vec{e}_\varphi=-\sin\varphi\ \vec{e}_x+\cos\varphi\ \vec{e}_y$$

$$\text{grad}\ \Phi=\left(\frac{\partial\Phi}{\partial\varrho}\frac{\partial\varrho}{\partial x}+\frac{\partial\Phi}{\partial\varphi}\frac{\partial\varphi}{\partial x}+\frac{\partial\Phi}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{=0}\right)\vec{e}_x+\left(\frac{\partial\Phi}{\partial\varrho}\frac{\partial\varrho}{\partial y}+\frac{\partial\Phi}{\partial\varphi}\frac{\partial\varphi}{\partial y}+\frac{\partial\Phi}{\partial z}\frac{\partial z}{\partial y}\right)\vec{e}_y+\frac{\partial\Phi}{\partial}$$

$$\frac{\partial\varrho}{\partial x}=\frac{x}{\varrho}=\cos\varphi$$

$$\frac{\partial\varphi}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2}=\frac{-\varrho\sin\varphi}{\varrho^2}=\frac{-\sin\varphi}{\varrho}$$

$$\text{grad}\,\Phi=\frac{\partial\Phi}{\partial\varrho}\vec{e}_\varrho+\frac{\partial\Phi}{\partial\varphi}\frac{1}{\varrho}\vec{e}_\varphi+\frac{\partial\Phi}{\partial z}\vec{e}_z$$

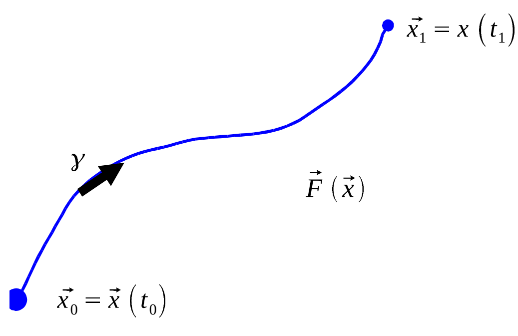
$$\text{grad}\ \Phi=\vec{e}_x\frac{\partial\Phi}{\partial x}+\vec{e}_y\frac{\partial\Phi}{\partial y}+\vec{e}_z\frac{\partial\Phi}{\partial z}=\frac{\partial\Phi}{\partial\varrho}\vec{e}_\varrho+\frac{\partial\Phi}{\partial\varphi}\frac{1}{\varphi}\vec{e}_\varphi+\frac{\partial\Phi}{\partial z}\vec{e}_z\ ;$$

$$\text{div}\ \vec{v}=\vec{\nabla}\cdot\vec{v}=\frac{\partial v_i}{\partial x_i}\equiv\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}$$

$$\vec{\nabla}=\vec{e}_x\frac{\partial}{\partial x}+\vec{e}_y\frac{\partial}{\partial y}+\vec{e}_z\frac{\partial}{\partial z}\ ;$$

$$\text{curl}\ \vec{v}=\vec{\nabla}\times\vec{v}=\begin{vmatrix}\vec{e}_1&\vec{e}_2&\vec{e}_3\\\frac{\partial}{\partial x_1}&\frac{\partial}{\partial x_2}&\frac{\partial}{\partial x_3}\\v_1&v_2&v_3\end{vmatrix}=\vec{e}_1\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right)-\vec{e}_2\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right)+\vec{e}_3\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v}{\partial x}\right.$$

1.2.1 Line integrals



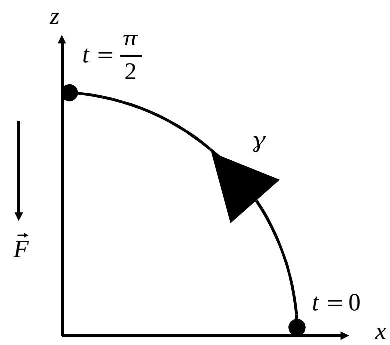
$$\int_\gamma\vec{F}\left(\vec{x}\right)\cdot\text{d}\ \vec{x}\ =\text{line integral of }\ \vec{F}\ \text{ along }\ \gamma$$

$$\text{d}\ \vec{x}=\frac{\text{d}\ \vec{x}}{\text{d}\ t}\ \text{d}\ t=\dot{\vec{x}}\ \text{d}\ t$$

$$\int_\gamma \vec{F} \cdot \mathrm{d}\,\vec{x} = \int_{t_0}^{t} F_x\,\dot{x}\,\mathrm{d}\,t + \int_{t_0}^{t} F_y\,\dot{y}\,\mathrm{d}\,t + \int_{t_0}^{t} F_z\,\dot{z}\,\mathrm{d}\,t$$

$$F_z\left(\vec{x}\left(t\right)\right)\,\dot{z}\left(t\right)$$

Example:



$$\vec{F} = -g\,\vec{e}_z\,;$$

$$\vec{x}\left(t\right) = \begin{pmatrix} a\,\cos\,t \\ 0 \\ a\,\sin\,t \end{pmatrix}$$

$$\dot{\vec{x}}\left(t\right) = \begin{pmatrix} -a\,\sin\,t \\ 0 \\ a\,\cos\,t \end{pmatrix}$$

$$\vec{F} \cdot \dot{\vec{x}} = -g\;a\;\cos\;t$$

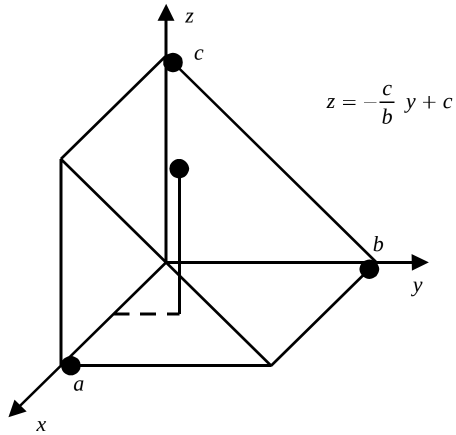
$$\int \vec{F} \cdot \mathrm{d}\,\vec{x} = -a - g\int_0^{\frac{\pi}{2}} \cos\,t\,\mathrm{d}\,t = -a\;g\,\cdot$$

1.2.2 Multiple integrals

Body with mass density:

$$\mu = \lim_{\Delta\,V \to 0} \frac{\Delta\,M}{\Delta\,V} = \mu_0$$

Specifically, **wedge**: Calculate M :



$$M = \iiint_{\text{wedge}} \mu(x, y, z)\, dx\, dy\, dz = \mu_0 \int_0^a dx \int_0^b dy \int_0^{-\frac{c}{b}y + c} dz \cdot 1$$

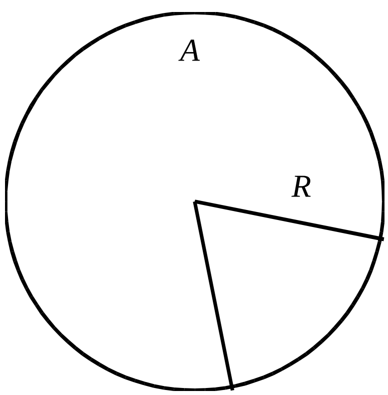
$$= \mu_0 \int_0^a dx \int_0^b dy \left(-\frac{c}{b}\ y + c\right) = \mu_0 \int_0^a dx \left(-\frac{c}{b}\ \frac{y^2}{2} + c\ y\right)_0^b$$

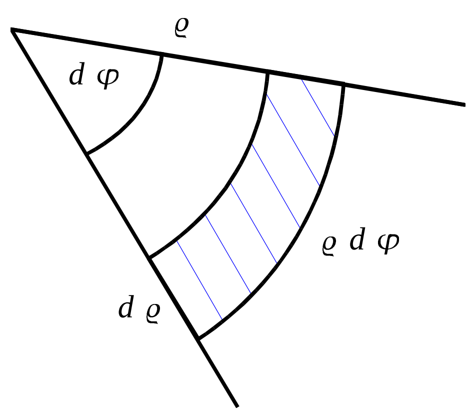
$$= \mu_0 \int_0^a dx \left(-\frac{c\ b}{2} + c\ b\right) = \mu_0\ \frac{a\ b\ c}{2};$$

Consider circle of constant **area density**:

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A} = \sigma_0$$

Calculate M:





$$M = \iint_{\text{circle}} \sigma\ dA$$

polar coord: ϱ, φ;

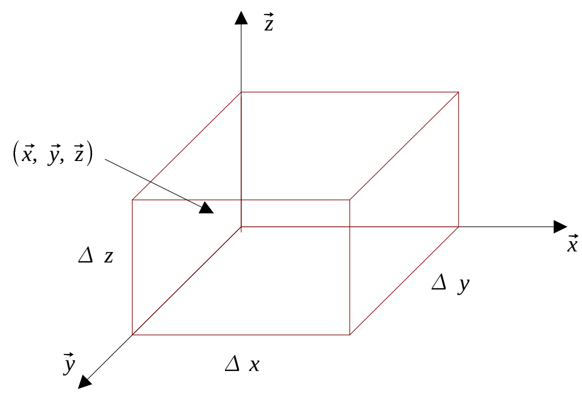
$$dA = dx\ dy = d\varrho \cdot \varrho\ d\varphi;$$

$$M = \sigma_0 \int_{\varrho=0}^{R} d\varrho\ \varrho \int_{\varphi=0}^{2\pi} d\varphi = \pi\ \sigma_0\ R^2$$

3 D: $d\varrho\ \varrho\ d\varphi\ dz = dV$

1.3 Gauss's Theorem

Momentum field where $\vec{u} = \varrho\,\vec{v}$ gives the momentum carried per unit volume (ϱ is the mass density, $\vec{v}$ velocity).



Mass that flows in and out of the box.
$\vec{v}$ is constant across each surface plane of the box.

I_x = Rate of flow *in* at x = mass that flows in per unit time at x = $\varrho\, v_x(x)\,\Delta y\,\Delta z$;

O_x = Rate of flow *out* at $x + \Delta x$ = $\varrho\, v_x(x+\Delta x)\,\Delta y\,\Delta z = \varrho\left(v_x(x) + \frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\,\Delta z$;

$O_x - I_x = \varrho\,\frac{\partial v_x}{\partial x}\,\Delta x\,\Delta y\,\Delta z$ = **Net** outward flow.

$O - I$ = net flow out of the box =

$$\varrho\underbrace{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)}_{\vec{\nabla}\cdot\vec{v}}\Delta x\,\Delta y\,\Delta z\,;$$

Net flow = density x component of velocity normal to the surface x

Area of surface = $\sum_{\text{surface of the box}} \varrho\,\vec{n}\cdot\vec{v}\,\Delta S$

Gauss's Theorem or **Divergence Theorem**

$$\int\int\int_V \vec{\nabla}\cdot\vec{A}\,\mathrm{d}V = \int\int_S \vec{n}\cdot\vec{A}\,\mathrm{d}S$$

$\vec{A}$: any vector field;
S : surface that bounds V;
$\vec{n}$ = unit vector normal to S, pointing outward

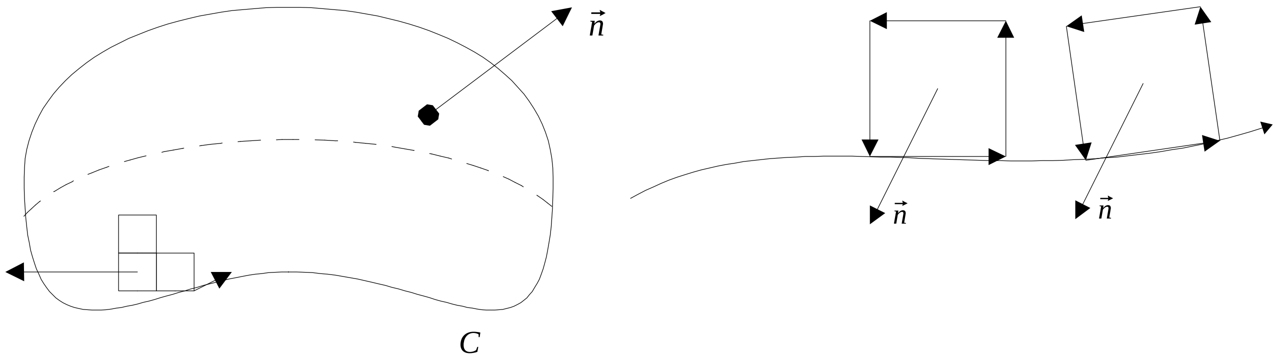
1.4 Stokes's Theorem

Surface integral over the curl of a vector field = line integral of that vector field along the boundary of the surface.

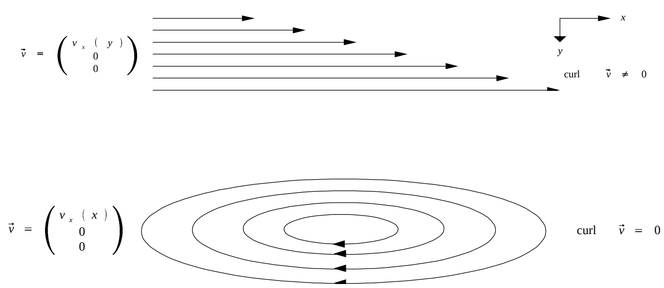
$$\int\int_S (\vec{\nabla} \times \vec{v}) \cdot \vec{n} \, \mathrm{d}S = \oint_C \vec{v} \cdot \mathrm{d}\vec{r}$$

$C = \partial S$ = boundary of S

$\vec{n}$ is the unit vector to the surface element $\mathrm{d}S$, pointing outwards



The direction used in evaluating the line integral is given by the right hand rule with respect to $\vec{n}$.



1.5 Ordinary differential equations

1.5.1 Basic definitions

An ODE of order n has the form

$$y^{(n)}(x) = f\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right).$$

Theorem 1: Its general solution has the form

$$y = y(x; c_1, \ldots, c_n)$$

where $c_1, \ldots, c_n$ are arbitrary real constants. For a complete and unique specification of the solution, n initial values must be specified.

$$y(x_0) = y_0, \; y'(x_0) = y_0^{(1)}, \; y''(x_0) = y_0^{(2)}, \ldots, \; y^{(n-1)}(x_0) = y_0^{(n-1)}$$

Initial Value Problem (IVP)

Theorem 2: Every ODE of n-th order can be transformed into a set of n first-order ODEs.

1) $y^{(n)}(x) = f\left(x, y'(x), \dots, y^{(n-1)}(x)\right)$
2) Introduce new variables: $y_1 = y$, $y_2 = y'$, ..., $y_n = y^{(n-1)}$
3) System of 1st order ODEs:
$y'_1 = y_2$, $y'_2 = y_2$, ..., $y'_{n-1} = y_n$, $y'_n = f(x, y_1, y_2, \dots, y_n)$.

Note: *A partial differential equation* $F\left(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, \dots, u_{x_n x_n}\right) = 0$

1.5.2 Special cases

Type 1: ODE that allows **separation of variables**

$\frac{\mathrm{d}y}{\mathrm{d}x} = y' = \frac{f(x)}{g(y)} \quad \Rightarrow \quad \int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x + C$ is general solution

Type 2: Linear first-order ODE

$y' + f(x)\,y = g(x)$

with

$M(x) = e^{\int f(x)\,\mathrm{d}x}$,

the general solution is

$$y(x) = \frac{1}{M(x)}\left[\int g(x)\,M(x)\,\mathrm{d}x + C\right]$$

Example: $y' + \frac{1}{x}\,y = x^2$

$$M(x) = e^{\int \frac{1}{x}\,\mathrm{d}x} = e^{\ln x} = x$$
$$y(x) = \frac{1}{x}\left[\int x^3\,\mathrm{d}x + C\right] = \frac{x^3}{4} + \frac{C}{x};$$

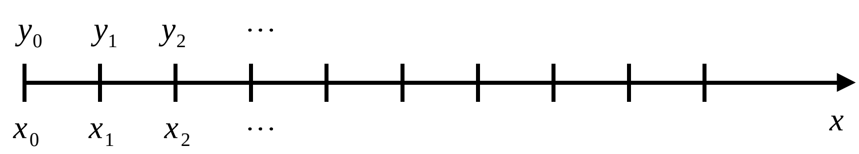
1.5.3 Numerical solution for ODEs

IVP: $y' = f(x, y)$; $y(x_0) = y_0$; $\Rightarrow$ $y(x) = ?$

By discretization: $h > 0$

convert x-space $\rightarrow$ coarse-grained space

$x_0, x_0 + h = x_1, x_1 + h = x_2, \dots$



The so-called „**forward Euler method**“ takes solution at $x_n \rightarrow x_{n+1}$:

$y_{n+1} = y_n + h \cdot f(x_n, y_n)$

Start: $n = 0 \dots (x_n, y_n) = y(x)\big|_n$

2. Principles of Dynamics

- Motion of objects acted upon by external forces.
- Point particles, characterized by $\vec{x} = \vec{x}(t)$ with $\vec{v}(t) = \dot{\vec{x}}(t)$
- t „time“ – before-after, assume that
  $\left(\vec{x}_i(t), \vec{v}_i(t)\right)_{i=1,\ldots,N}$
  can be measured *simultaneously*
  Assumption: $|\vec{v}_i| \ll c$
- Assume that $\vec{x}$ and $\vec{v}$ can be measured for each point particle: $\Rightarrow$ out of quantum mechanics

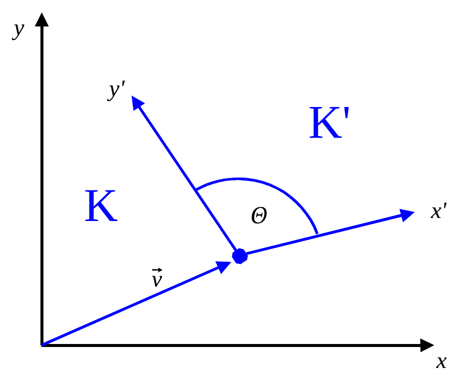
2.1 Newton's laws

Physical space: 3 D, Euclidean metric

> **Principle 1 (Newton's first law)**: There exist certain frames of reference, called **inertial**, with prop's:
> A. Every isolated particle moves in a straight line in such a frame.
> B. Every isolated particle moves at constant velocity in this frame.

Consequences

- The existence of one inertial frame implies the existence of many more.
- Due to A, they cannot rotate with respect to each other.



- Transformations: **Galileian**

> **Principle 2 (3rd Newton's law)**:
> A. **Conservation of Momentum**:
> Consider 2 particles, isolated from all other matter, from an inertial frame, with velocities $\vec{v}_j(t)$, $j = 1, 2, \ldots$.
> Then there exists a constant $\mu_{12} > 0$ and a constant vector $\vec{K}$ independent of time such that
> $\vec{v}_1(t) + \mu_{12}\vec{v}_2(t) = \vec{K}$.
> μ_{12} is universal (i.e. neither depends on inertial frame nor on particular motion).
>
> B. **Existence of Mass**:
> Bringing in a third particle such that watch 1-3 and then 2-3.

$$\vec{v}_3(t) + \mu_{31}\,\vec{v}_1(t) = \vec{M}$$
$$\vec{v}_2(t) + \mu_{23}\,\vec{v}_3(t) = \vec{L}$$
Then:
$$\mu_{12}\,\mu_{23}\,\mu_{31} = 1 \quad (3)$$

(3) $\Rightarrow \quad \exists\, m_i > 0\ (i = 1, 2, 3)$ such that (1) + (2):
$$m_1\,\vec{v}_1(t) + m_2\,\vec{v}_2(t) = \vec{P}_{12}$$
$$m_2\,\vec{v}_2(t) + m_3\,\vec{v}_3(t) = \vec{P}_{23}$$
$$m_3\,\vec{v}_3(t) + m_1\,\vec{v}_1(t) = \vec{P}_{31}$$
$\vec{P}_{ij}$, called **momenta** (Impulse), is constant.

- Masses are *not* unique, only ratios μ of masses is unique. Standard must be chosen (e.g. $1\ \mathrm{cm}^3$ of water at $4\,°\mathrm{C}$ and atmospheric pressure).
- $\frac{\mathrm{d}}{\mathrm{d}\,t}$ (4)
  $\Rightarrow \quad m_i\,\vec{a}_i + m_j\,\vec{a}_j = 0 \quad (i \neq j)$

Define the force acting on a particle by
$$\vec{F} = m\,\vec{a} = m\,\frac{\mathrm{d}^2\,\vec{x}}{\mathrm{d}\,t^2}\,.$$

Newton's second law: Particles 1 and 2: $\vec{a}_1$ of particle 1 arises as a result of the interaction between 1-2. One says there is a **force** $\vec{F}_{12} = m_1\,\vec{a}_1$ on particle 1 due to the presence of particle 2 or by particle 2. Also, $\vec{F}_{21} = m_2\,\vec{a}_2$ on particle 2 by part 1.
$$\vec{F}_{12} + \vec{F}_{21} = 0\,.$$
To every **action** there is an equal and opposite **reaction**.
Newton's 3rd law:
$$\vec{F} = m\,\frac{\mathrm{d}^2\,\vec{x}}{\mathrm{d}\,t^2}\,.$$

Forces satisfy the **superposition principle**. The total force on a particle can be found by adding contributions from different agents.

Remark*: For gravitation,* $\vec{a}$ *is known a priori.*

2.2 Stability of solutions

$\vec{F}$ is given $\vec{x}$, $\dot{\vec{x}}$, t

Set of 3 2nd-order ODEs
$$\vec{F}(\vec{x}, \dot{\vec{x}}, t) = m\,\ddot{\vec{x}}$$
$$x(t_0),\ \dot{\vec{x}}(t_0)$$

Stability of solutions

$\vec{x}_1(t_0)$ and $\vec{x}_2(t_0) = \vec{x}_1(t_0) + \delta\,\vec{x}(t_0)$

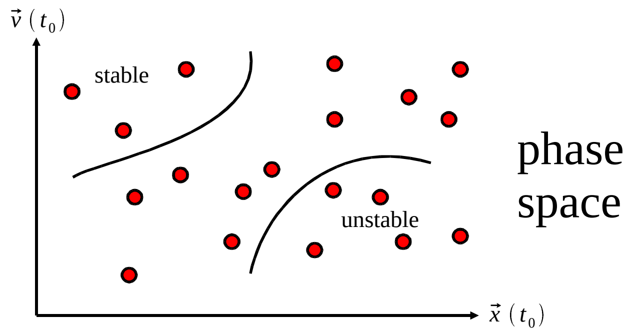
$D\vec{x}(t) = |\delta\vec{x}(t_0)| < \varepsilon$ and assume $\dot{\vec{x}}_1(t_0) = \dot{\vec{x}}_2(t_0)$

Then solutions are **stable** if

$$\lim_{t\to\infty} D\vec{x}(t) = O(D x(t_0))$$

and **unstable** if

$$\lim_{t\to\infty} D\vec{x}(t) = \infty$$



Regions, stable + unstable solutions are mixed together tightly. Such a system is called **chaotic** (in given portion of phase space), **exhibits chaos**.

2.3 One-particle dynamical variables

$\vec{F} = m\vec{a} \Rightarrow \vec{x}(t)$

Symmetry $\Rightarrow$ $f(\vec{x}, \dot{\vec{x}}, t)$
dynamical variables characterize motion

2.3.1 Momentum

$$\vec{p} = m\vec{v} \quad \Rightarrow \quad \vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p}$$

$$\vec{F} = 0 \Rightarrow \vec{p} = \text{conserved}$$

If total force $= 0$, $\vec{p}$ is an constant of motion.

Note: $m = m(t), \vec{v} = \vec{v}(t) \Rightarrow \vec{p}$

Assumption: All measurements are performed in inertial systems.

2.3.2 Angular momentum

$\vec{L}$ about some point $\vec{S}$:

$\vec{L} = \vec{x} \times \vec{p}$

where $\vec{x}$ is measured from the point $\vec{S}$.

$\vec{S}$ must be **inertial point**, moves at $\vec{v} = \text{const}$ in any inertial frame.

$$\dot{\vec{L}} = \frac{\mathrm{d}}{\mathrm{d}\,t}(\vec{x} \times \vec{p}) = \underbrace{\vec{v} \times \vec{p}}_{=0} + \vec{x} \times \underbrace{\dot{\vec{p}}}_{=\vec{F}};$$

$$\dot{\vec{L}} = \vec{x} \times \vec{F} = \vec{N}$$

torque (Drehmoment) about $\vec{S}$

$$\dot{\vec{p}} = \vec{F}$$

$$\vec{N} = 0 \Rightarrow \vec{L} = \mathrm{const}$$

2.3.3 Energy and work

Often, $\vec{F}(\vec{x})$

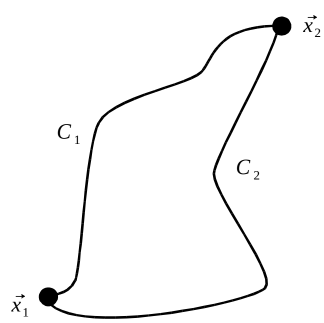
$$\int_{C\,:\,\vec{x}(t_0)}^{\vec{x}(t)} \vec{F}(\vec{x}) \cdot \mathrm{d}\,\vec{x} = \int_{t_0}^{t} \vec{F}(\vec{x}) \cdot \dot{\vec{x}}\,\mathrm{d}\,t = m \int_{t_0}^{t} \ddot{\vec{x}} \cdot \dot{\vec{x}}\,\mathrm{d}\,t = \frac{1}{2} m \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\dot{x}^2\right) \mathrm{d}\,t = \frac{1}{2} m\,v^2(t) - \frac{1}{2} m\,v^2(t_0)$$

;

T **kinetic energy**

Kinetic energy $T = 1/2\,m\,v^2$ and the work W_C done by the force along C:
$W_C = \int_C \vec{F} \cdot \mathrm{d}\,\vec{x}$.

If W is not path-dependent, the force is called **conservative force**:



Conservative: $\Rightarrow$

$$\int_{C_1} \vec{F} \cdot \mathrm{d}\,\vec{x} = \int_{C_2} \vec{F} \cdot \mathrm{d}\,\vec{x};$$

$$\int_{C_1} \vec{F} \cdot \mathrm{d}\,\vec{x} - \int_{C_2} \vec{F} \cdot \mathrm{d}\,\vec{x} = \oint \vec{F} \cdot \mathrm{d}\,\vec{x} = 0.$$

If $\vec{F}$ conservative,

$$\oint \vec{F} \cdot \mathrm{d}\,\vec{x} = 0$$

around *any* closed path, for $\vec{x}_1$, $\vec{x}_2$ arbitrary.

$$\oint_{C=\partial\Sigma} \vec{F} \cdot \mathrm{d}\,\vec{x} = \int_{\Sigma} \left(\vec{\nabla} \times \vec{F}\right) \cdot \mathrm{d}\,\vec{S}$$

Σ is any smooth surface bounded by the closed path $C = \partial\,\Sigma$.

$$\vec{\nabla} \times \vec{F} = 0 \Leftrightarrow F = \text{conservative}$$

curl grad $V \equiv 0$;
$\vec{\nabla} \times (\vec{\nabla} V) \equiv 0$;
$\vec{F}(\vec{x}) = -\vec{\nabla} V(\vec{x})$;

$V(\vec{x})$ **potential energy**

Note: $\vec{V}$ *is not unique and defined only up to an additive constant.*

$$W_C = W(\vec{x}, \vec{x}_0) = \int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}\,') \cdot \mathrm{d}\vec{x}\,' = -\int_{\vec{x}_0}^{\vec{x}} \vec{\nabla} V(\vec{x}\,') \cdot \mathrm{d}\vec{x}\,' = V(\vec{x}_0) - V(\vec{x}) \equiv V_0 - V$$

$V_0 - V = T - T_0$; $V + T = V_0 + T_0 = E = \text{const}$;

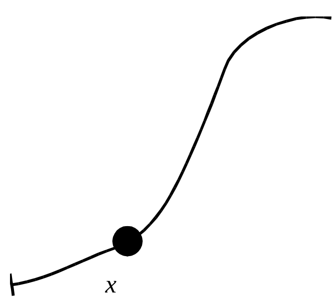
E **total mechanical energy**

$\Rightarrow$ conservation of energy for conservative forces

Note: E *is also defined only up to an additive constant; usual:* $V(\vec{x}_0) = 0$.

2.3.4 Application to 1-D motion

Number of functions of time needed to describe fully the system is called **number of degrees of freedom** or **number of freedoms**.



$\Rightarrow$ one-freedom system, can be reduced to **quadrature**

x : distance along line
F along line $= F(x)$ $\Rightarrow$ necessarily conservative

$$\Rightarrow \quad m\ddot{x} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x} .$$

The energy

$$\frac{1}{2} m \dot{x}^2 + V(x) = E \quad (*)$$

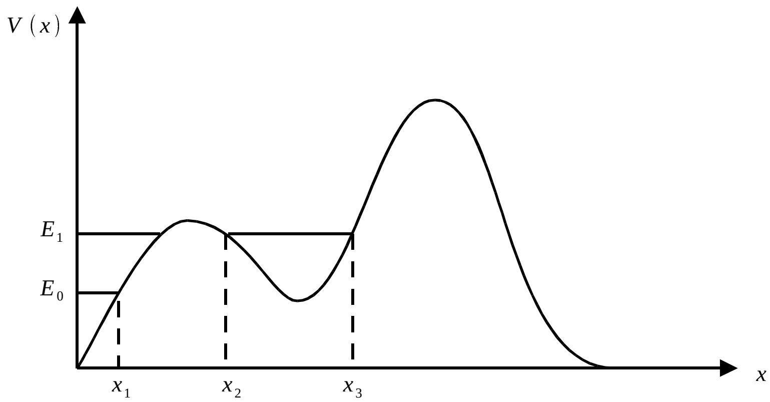
$1/2\, m\, \dot{x}^2 + V$ **energy first-integral**

$$\frac{1}{\dot{x}} = \frac{\mathrm{d}\,t}{\mathrm{d}\,x} \quad \Rightarrow \ (*) \ \Rightarrow$$

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{\mathrm{d}\,x}{\sqrt{E - V(x)}}$$

$$x_0 = x(t_0)\,,\ x = x(t)$$

$$\Rightarrow\ x = x(t - t_0\,,\ E)$$



$$\underbrace{\frac{1}{2}\, m\, \dot{x}^2}_{>0} + V(x) = E$$

Particle cannot enter region $E = V(x) < 0$

– $E - V(x) > 0:\ E = E_0$
  If particle moves:
  $V \uparrow \Rightarrow \frac{1}{2}\, m\, \dot{x}\,\acute{}2 \downarrow$
  until $E - V(x) = 0 : \dot{x} = 0 \Rightarrow x_1$.
  At $x_1:\ F(x_1) = -\left.\frac{\mathrm{d}\,V}{\mathrm{d}\,x}\right|_{x_1} < 0$
  $\Rightarrow$ particle gets accelerated to the left

– $E_1 : x_2 < x < x_3$ motion *bounded*
  Period:
  $$P = \sqrt{2\, m} \int_{x_2}^{x_3} \frac{\mathrm{d}\,x}{\sqrt{E - V(x)}}$$

– $\frac{\mathrm{d}\,V}{\mathrm{d}\,x} = 0$ **equilibration points**
  can be *stable* (minima) or *unstable* (maxima)

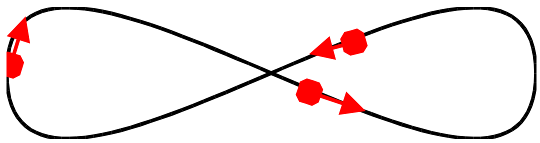
2.4 Many-particle system

2.4.1 Superposition principle

$N > 1$ particles may interact with one another and also are acted on by external forces.

Only 2-particle problem can be solved generally and analytically.

A. Chenciner and R. Montgomery: 3 particles, equal mass, interact via gravitation; discovered the first exact, stable and periodic nontrivial solution of 3-body-problem



Superposition principle: Total force acting on a particle is a vector sum of all the forces exerted on it by each of the other particles in the system (the **internal forces**) plus all of the **external forces**.

Superposition principle implies that all forces can be analyzed in terms of *2-particle interactions*.

A , B , C . The force on A is the sum of the force it would feel if only B were present and the force it would feel if only C were present.

This superpositional principle is an additional assumption to Newtons' laws, it is a result of experimental observation.

$\Rightarrow$ Assume two-body forces

2.4.2 Center of Mass

$\vec{F}_{ij} =$ force exerted on i-th particle by the j-th particle. $\Rightarrow \vec{F}_{ii} = 0$

Newton's 3rd law $\Rightarrow \quad \vec{F}_{ij} = -\vec{F}_{ji}$

***Note**: Abandon the summation convention in this section!*

$$m_i \ddot{\vec{x}}_i = \vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ij}$$

$$\sum_i m_i \ddot{\vec{x}}_i = \underbrace{\sum_i \vec{F}_i}_{=\vec{F}} + \underbrace{\sum_{i,j} \vec{F}_{ij}}_{=0}$$

$\vec{F}$ total external force = sum over only external forces

$$M = \sum_i m_i \quad \Rightarrow$$

$$\vec{F} = M \left(\frac{1}{M} \sum_i m_i \ddot{\vec{x}}_i \right) \equiv M \ddot{\vec{X}} ;$$

$$\vec{X} \equiv \frac{1}{M} \sum_{i=1}^{N} m_i \vec{x}_i \quad \textbf{center of mass}$$

Center-of-mass theorem: The center of mass of N-particle system moves as though the total mass were concentrated there and were acted upon by the total external force.

2.4.3 Momentum

Total momentum

$$\vec{p} = \sum_i m_i \dot{\vec{x}}_i = M \dot{\vec{X}}$$

$$\Rightarrow \quad \vec{F} = \dot{\vec{P}}$$

$$\text{if } F_{ext} = 0 \Rightarrow \vec{P} = \text{const}$$

$$\Rightarrow M \text{ constant} \Rightarrow \dot{\vec{X}} = \text{const} \Rightarrow \vec{X} = \vec{x}_0 + \underbrace{\vec{C}}_{=\frac{\vec{P}_0}{m}} t$$

$\Rightarrow$ 6 conserved dynamical variables associated with center of mass

2.4.4 Energy

$\vec{y}_i = \vec{x}_i - \vec{X}$ position of the i-th particle *relative* to the center of mass

$$T = \frac{1}{2} \sum_i m_i \dot{\vec{x}}_i^{\,2} = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} \sum_i m_i \dot{y}_i^2$$

Total work done on all particles: initial configuration $\{0\}$ ($3N$ coordinates) $\rightarrow$ final configuration $\{f\}$ ($3N$ momenta)

$$\sum_i \int_{\vec{x}_{i\,0}}^{\vec{x}_{i\,f}} \left(\vec{F}_i + \sum_j \vec{F}_{ij} \right) \cdot \mathrm{d}\,\vec{x}_i = \underbrace{\sum_i \int_{\vec{x}_{i\,0}}^{\vec{x}_{i\,f}} m_i \ddot{\vec{x}}_i \cdot \mathrm{d}\,\vec{x}}_{T_f - T_i}$$

r.h.s.: $T_f - T_0$

Assume: $\vec{F}_i = -\vec{\nabla}_i V_i(\vec{x}_i)$; $\vec{F}_{ij} = -\vec{\nabla}_i V_{ij}\left(|\vec{x}_i - \vec{x}_j|\right)$.

$$\Rightarrow \quad \vec{F}_{ji} = -\vec{F}_{ij}$$

$$\sum_i \int_{\vec{x}_{i\,0}}^{\vec{x}_{i\,f}} \vec{F}_i \,\mathrm{d}\,\vec{x}_i = \sum_i \left(V_i^0 - V_i^f \right) = V_{ext}^0 - V_{ext}^f \,;$$

$$\int_{\vec{x}_{i\,0}}^{\vec{x}_{i\,f}} \sum_i j\, \vec{F}_{ij} \cdot \mathrm{d}\,\vec{x}_i = \frac{1}{2} \left(\sum_{i\,j} \vec{F}_{i\,j} \cdot \mathrm{d}\,\vec{x}_i + \sum_{i\,j} \underbrace{\vec{F}_{j\,i}}_{-\vec{F}_{i\,j}} \cdot \mathrm{d}\,\vec{x}_j \right) = \frac{1}{2} \int_{\vec{x}_{i\,0}}^{\vec{x}_{i\,f}} \sum_{i\,j} \vec{F}_{i\,j} \cdot \mathrm{d}\,\underbrace{(\vec{x}_i - \vec{x}_j)}_{\vec{x}_{i\,j}}$$

$$= \frac{1}{2} \int_{\vec{x}_{i\,0}}^{\vec{x}_{i\,f}} \sum_{i\,j} \vec{F}_{i\,j} \cdot \mathrm{d}\,\vec{x}_{i\,j} = \frac{1}{2} \sum_{i\,j} \left(V_{i\,j}^0 - V_{i\,j}^f \right) = V_{\text{int}}^0 - V_{\text{int}}^f$$

$$\Rightarrow \quad \left(V_{\text{ext}} + V_{\text{int}} + T \right)^0 = \left(V_{\text{ext}} + V_{\text{int}} + T \right)^f = E$$

***Note**: We have assumed that the internal forces can be derived from*

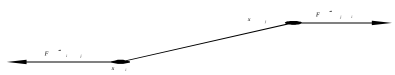
$$V_{i\,j}(x)\,,\; x = |\vec{x}_i - \vec{x}_j|$$

$$\Rightarrow \vec{F}_{i\,j} = -\frac{\mathrm{d}\,V_{i\,j}(x)}{\mathrm{d}\,x} \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}$$

Central force acting along the connecting line between $\vec{x}_i$ and $\vec{x}_j$. This implies:

$$\left(\vec{x}_i - \vec{x}_j\right) \times \vec{F}_{i\,j} = 0\,;$$

This is a *supplement* to the 3rd Newton's principle.



2.4.5 Angular momentum

Inertial reference system: Compute the total angular momentum about the origin.

$$\vec{L} = \sum_i m_i\, \vec{x}_i \times \dot{\vec{x}}_i = \sum_i \vec{x}_i \times \vec{p}_i$$

The total angular momentum about the center of mass is

$$\vec{L}_c = \sum_i m_i\, \vec{y}_i \times \dot{\vec{y}}_i\ ;\ \vec{y}_i = \vec{x}_i - \vec{X}\,.$$

$$\vec{L} = \sum_i m_i \left(\vec{X}\;\; \vec{y}_i\right) \times \left(\dot{\vec{X}}\;\; \dot{\vec{y}}_i\right) = \sum_i m_i\, \vec{X} \times \dot{\vec{X}} + \sum_i m_i\, \vec{y}_i \times \dot{\vec{y}}_i + \underbrace{\sum_i m_i\, \vec{y}_i \times \dot{\vec{X}}}_{=0} + \underbrace{\sum_i m_i\, \vec{X} \times \dot{\vec{y}}_i}_{=0}$$

(3rd sum: position of center of mass relative to itself; 4th sum: velocity of center of mass relative to itself)

$$\vec{L} = \vec{L}_c + M\, \vec{X} \times \dot{\vec{X}}$$

The total angular momentum about the origin of an inertial system = sum of the angular momenta of the total mass as though all the mass were concentrated at the center of mass + inertial angular momentum = angular momentum about the center of mass.

Question: Does $N = \dot{\vec{L}}$ hold also for many-particle system?

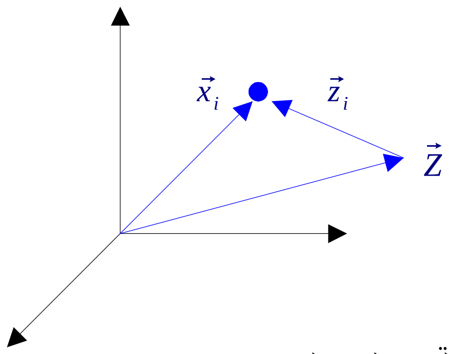
Supplement to 3rd Newton's law about $\vec{F}_{i\,j}$ guarantees that the internal torques do not contribute to the change in the total angular momentum.

$\Rightarrow$ Need to consider solely the externally applied torque.

The external forces apply a total torque about an arbitrary (moving) point $\vec{Z}$:

$$\vec{N}_Z = \sum_i \vec{z}_i \times \vec{F}_i = \sum_i m_i\, \vec{z}_i \times \ddot{\vec{x}}_i\,,$$

where $\vec{x}_i$ = position of i-th particle in an inertial frame; $\vec{z}_i = \vec{x}_i - \vec{Z}$ is its position relative to $\vec{Z}$.



$$\vec{N}_Z = \sum_i m_i\, \vec{z}_i \times (\ddot{\vec{z}}_i + \ddot{\vec{Z}}) = \frac{d}{dt} \sum_i m_i\, \vec{z}_i \times \dot{\vec{z}}_i + M\,(\vec{X} - \vec{Z}) \times \ddot{\vec{Z}} = \dot{\vec{L}}_Z + M\,(\vec{X} - \vec{Z}) \times \ddot{\vec{Z}};$$

$$\vec{N}_Z = \dot{\vec{L}}_Z \text{ if } \vec{X} - \vec{Z} \parallel \ddot{\vec{Z}}$$

special cases are:

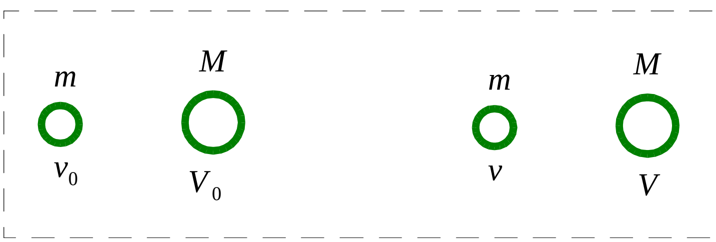
a) $\ddot{\vec{Z}} = 0 \Rightarrow \vec{Z}$ is moving at constant velocity relative to the origin of the inertial frame

b) $\vec{X} = \vec{Z} \Rightarrow \vec{Z}$ is chosen as the center of mass of the system.

[http://www.ai.mit.edu/people/wessler/halo/mont.html]

2.5 Examples

2.5.1 One-dimensional collision between point particles



Conservation of total momentum:

$$m\, v_0 + M\, V_0 = m\, v + M\, V$$

$$m\,(v - v_0) = M\,(V - V_0) \quad (*)$$

$$x_{CM} = \frac{m\, x_0 + M\, X_0}{m + M}$$

$$\frac{m\, v_0 + M\, V_0}{m + M} = \frac{m\, v + M\, V}{m + M}$$

Elastic collision – energy conserved:

$$\frac{m\, v_0^2}{2} + \frac{M\, V_0^2}{2} = \frac{m\, v^2}{2} + \frac{M\, V^2}{2};$$

$$m\,(v^2 - v_0^2) = M\,(V_0^2 - V^2)$$

(*) $\rightarrow \quad v + v_0 = V_0 + V$ (**)

(*) & (**) $\Rightarrow$

$$V = \frac{M - m}{m + M} V_0 + \frac{2\ m}{m + M} v_0 ;$$

$$v = \frac{m - M}{m + M} v_0 + \frac{2\ M}{m + M} V_0 ;$$

Fully inelastic collision:

$$v = M$$

Conservation of momentum:

$$m\ v_0 + M\ V_0 = (m + M)\ v$$

$$v = \frac{m\ v_0 + M\ V_0}{m + M} ;$$

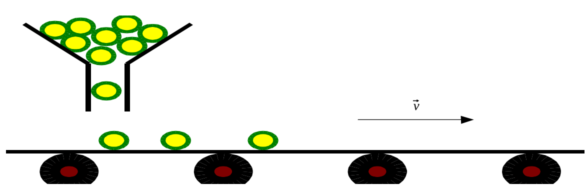
$$E_{kin,\ initial} = \frac{m}{2} v_0^2 + \frac{M}{2} V_0^2 ;$$

$$E_{kin,\ final} = \frac{m + M}{2} v^2 = \frac{1}{2} \frac{(m\ v_0 + M\ V_0)^2}{m + M} ;$$

$$\Delta\ E_{loss}^{kin} = E_{kin,\ initial} - E_{kin,\ final} = \frac{\mu}{2} (v_0 - V_0)^2 ;$$

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M} ;$$

2.5.2 Example: Conveyor belt



m : mass of the material (on the belt)
M : mass of the belt
v : velocity of the system

Force?
Which force is needed to keep a conveyor belt moving at constant speed of $5\ \mathrm{m/s}$, if the material gets dumped onto the belt at constant rate $\mathrm{d}\ m/\mathrm{d}\ t = 100\ \mathrm{kg/s}$?

The total momentum is

$$P = (m + M)\ v ;$$

$$F = \frac{\mathrm{d}\ P}{\mathrm{d}\ t} = v \frac{\mathrm{d}\ m}{\mathrm{d}\ t} = 5 \cdot 100\ \mathrm{N} ;$$

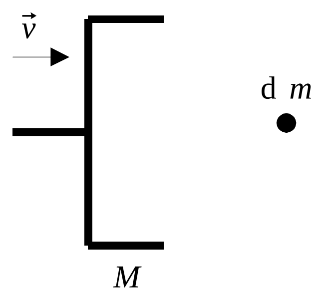
The power, i.e. the energy transfer per unit of time

$$\frac{d\ W}{d\ t} = \vec{F} \cdot \frac{d\ \vec{x}}{d\ t} = F\ v = v^2 \frac{d\ m}{d\ t} = \frac{d}{d\ t}(m\ v^2) = \frac{d}{d\ t}(m\ v^2 + M\ v^2) = \frac{d}{d\ t}\left((m+M)\ v^2\right) = \frac{d}{d\ t}(2$$

;

2.5.3 Example: snow plow

(british: plough)



$$d\ E_{kin} = \frac{d\ m\ M}{2\ (M\ r\ d\ m)}(0 - v)^2 = \frac{1}{2}\ d\ m\ v^2$$

$$v_{d\ m} = v_M = \frac{M\ v}{M + d\ m} \approx v\left(1 - \frac{d\ m}{M}\right) + \dots$$

$$E_{initial} = \frac{1}{2}\ M\ v^2$$

$$E_{final} = \frac{1}{2}\ (M + d\ m)\ v^2$$

i.e.:

$$d\ E_{kin} = E_{kin,\ final} - E_{kin,\ initial} = \frac{1}{2}\ d\ m\ v^2$$

$$\frac{d\ E}{d\ t} = 2 \cdot \frac{1}{2} \frac{d\ m}{d\ t}\ v^2$$

Initial momentum $p = m\ v$
Final momentum $p' = (M + d\ m)\ v$

$$F = \frac{d\ p}{d\ t} = \frac{d\ m}{d\ t}\ v\ ;$$

($\Delta\ p = d\ m\ v\ ;\ \Delta\ P = F\ \Delta\ t$)

$$\frac{d\ W}{d\ t} = F \cdot v = \frac{d}{d\ t}(m\ v^2)\ ;$$

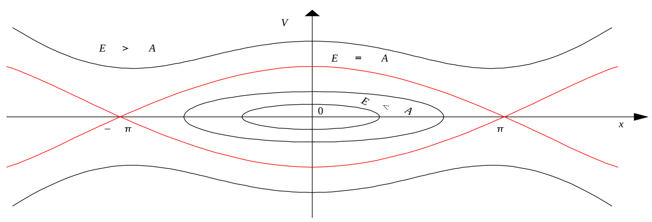
2.6 Velocity phase space and phase portraits

N particles
$6\ N$ variables
$3\ N$ coordinates
$3\ N$ velocities

$$v(x) = -A \cos x ; \quad A > 0 , -\infty < x < \infty ;$$
$$E = \frac{1}{2} m \dot{x}^2 + v(x) = \frac{1}{2} m \dot{x}^2 - A \cos x = \frac{1}{2} m v^2 - A \cos x ;$$

For $E > A$, the motion takes place entirely within one of the wells

$$p_{Period} = \sqrt{2m} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{t - v(x)}} = \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{E + A \cos x}} ;$$
$$x_{1,2} = \pm \cos^{-1}\left(\frac{E}{A}\right)$$
$$F = -\frac{\mathrm{d}v}{\mathrm{d}x} ;$$
$$E = A ;$$
$$v = \pm \sqrt{\frac{2}{m} (E + A \cos x)}$$



(v , x) **velocity phase space**

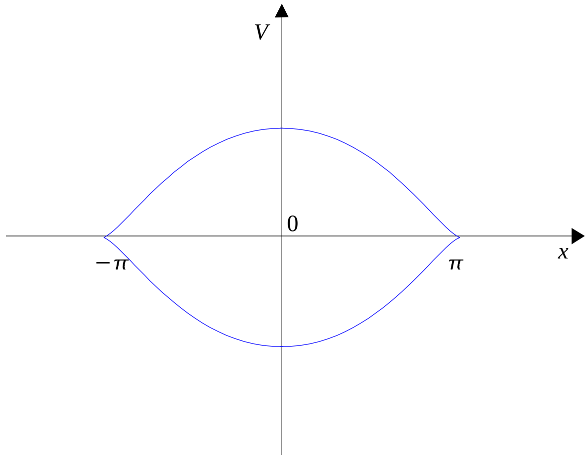
Such a graph is called **phase portrait** of the dynamical system

x near 0:

$$v(x) = v(0) + x\, v'(0) + \frac{1}{2} x^2 v''(0) + \ldots \approx A\left(-1 + \frac{1}{2} x^2 + \ldots\right) ;$$
$$\frac{m v^2}{2(E + A)} + \frac{A x^2}{2(E + A)} = 1 ;$$

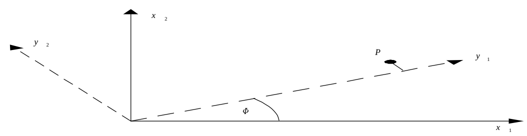
Separatrix: the curve corresponding to $E = A$ separates the bounded motion from the unbounded one.

Separatrix consists of 4 orbits for $x \in [-\pi , \pi]$.



2.7 Rotating frames

Inertial frame



Rotation around $x_3 = y_3$

Inertial frame: (x_1, x_2, x_3)
Rotating frame: (y_1, y_2, y_3)

$$\frac{d\phi}{dt} = \omega = const$$

(ω : angular velocity)

$\vec{\omega}$ angular velocity vector
$|\vec{\omega}| = \omega$
$\vec{\omega} \parallel$ axis of rotation

$\phi = \omega t + \phi_0$, and we set $\phi_0 = 0$; $\phi = \omega t$

$$\begin{matrix} y_1 = \\ y_2 = \\ y_3 = \end{matrix} \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$y_1 = x_1 \cos \omega t - \sin \omega t \, x_2$$
$$y_2 = x_1 \sin \omega t + \cos \omega t \, x_2$$
$$y_3 = x_3$$

Free particle: $\ddot{\vec{x}} = 0$ in inertial frame

$$\dot{y}_1 = \dot{x}_1 \cos \omega t - \dot{x}_2 \sin \omega t - \omega \, y_2$$
$$\dot{y}_2 = \dot{x}_1 \sin \omega t + \dot{x}_2 \cos \omega t + \omega \, y_1$$
$$\dot{y}_3 = \dot{x}_3$$

$$\ddot{y}_1 , \ddot{y}_2 , \ddot{y}_3 = ?$$

Using $\ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0$:

$\ddot{y}_1 = \omega\,(-\dot{x}_1 \sin\,\omega\,t - \dot{x}_2 \cos\,\omega\,t) - \omega\,\dot{y}_2 = -\omega\,(\dot{y}_2 - \omega\,y_1) - \omega\,\dot{y}_2 = \omega^2\,y_1 - 2\,\omega\,\dot{y}_2$;

$\ddot{y}_2 = \omega\,(\dot{x}_1 \cos\,\omega\,t - \dot{x}_2 \sin\,\omega\,t) + \omega\,\dot{y}_1 = \omega\,(\dot{y}_1 + \omega\,y_2) + \omega\,\dot{y}_1 = \omega^2\,y_2 + 2\,\omega\,\dot{y}_1$;

$\ddot{y}_3 = 0$;

$$\ddot{\vec{y}} = \vec{\omega} \times (\vec{\omega} \times \vec{y}) + 2\,\vec{\omega} \times \dot{\vec{y}}$$

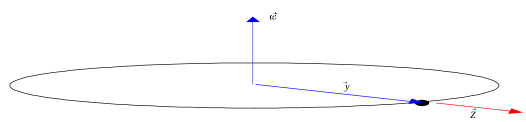
$\vec{\omega}$ is angular velocity vector, of magnitude ω , pointing along the rotation axis.

$$\vec{F} = -m\,\ddot{\vec{y}} = \vec{Z} + \vec{C}$$

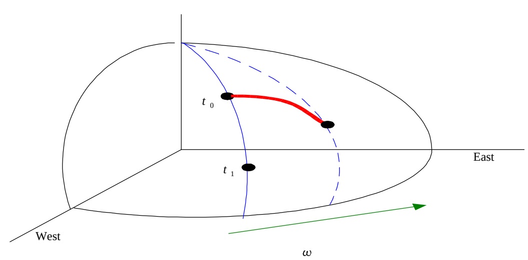
$\vec{Z} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{y})$ **centrifugal force**

$\vec{C} = -2\,m\,\vec{\omega} \times \dot{\vec{y}}$ **coriolis force**

„fictitious forces“



Rotating Earth: $|Z| \sim 0.3\ \%$ of the gravitational force at the equator



Plane: moves strictly along a meridian at constant height and speed in inertial frame

2.8 The Lorentz force

The electric field at $\vec{x}$ due to a point charge q_i at $\vec{x}_i$ is given by **Coulomb's law**:

$$\vec{E}\,(\vec{x}) = k\,q_i\,\frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}$$

$\vec{E}\,(\vec{x})$ is the force per unit charge acting at $\vec{x}$:

$\vec{F}\,(\vec{x}) = q_2\,\vec{E}\,(\vec{x})$

$k = (4\,\pi\,\varepsilon_0)^{-1}$,

$$\varepsilon_0 = 8.854 \cdot 10^{-12} \frac{\text{Farad}}{\text{meter}}$$

permittivity of free space

$$\vec{E}(\vec{x}) = -\vec{\nabla} \phi(\vec{x})$$

$$\phi(\vec{x}) = -k\, q_i \frac{1}{|\vec{x} - \vec{x}_i|}$$

A continuous charge distribution $\varrho(\vec{x})$

$$\vec{E}(\vec{x}) = k \int \varrho(\vec{x}') \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \mathrm{d}^3 x'$$

$$\phi(\vec{x}) = k \int \frac{\varrho(\vec{x}')}{|\vec{x} - \vec{x}'|} \mathrm{d}^3 x'$$

Maxwell's equations for static electric fields:

Theorem 1 (Gauss's law): $\varepsilon_0 \vec{\nabla} \cdot \vec{E}(\vec{x}) = \varepsilon_0 \operatorname{div} \vec{E}(\vec{x}) = \varrho(\vec{x})$

Theorem 2: $\vec{\nabla} \times \vec{E}(\vec{x}) = \operatorname{curl} \vec{E}(\vec{x}) = 0$

Proof:

Definition: The **Laplace operator** or **Laplacian** is defined by:

$$\Delta \equiv \vec{\nabla} \cdot \vec{\nabla} \equiv \operatorname{div} \operatorname{grad} \underset{\text{cartesian coordinates}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Definition: The δ**-function** is a special type of function, called functional or distribution, with following properties:

$$\delta(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$$

such that

$$\int f(\vec{x})\, \delta(\vec{x})\, \mathrm{d}^3 x = f(0)$$

for any continuous and integrable function $f(\vec{x})$.

$$\Delta\left(\frac{1}{r}\right),\ r = |\vec{x}| = \sqrt{x^2 + y^2 + z^2}\ ;$$

$$\Delta\left(\frac{1}{r}\right) = \vec{\nabla} \cdot \left(-\frac{\vec{r}}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{-z}{r^3}\right)$$

$$= \left(3\frac{x^2}{r^5} - \frac{1}{r^3}\right) + \left(3\frac{y^2}{r^5} - \frac{1}{r^3}\right) + \left(3\frac{z^3}{r^5} - \frac{1}{r^3}\right) = 0$$

for $r \neq 0$.

$r = 0$ case: Integrate over a small volume V containing the origin and apply Gauss's integral theorem:

$$\int_V \nabla^2 \left(\frac{1}{r}\right) \mathrm{d}^3 x = \int_V \vec{\nabla} \cdot \vec{\nabla} \left(\frac{1}{r}\right) \mathrm{d}^3 x \underset{\text{Gauss}}{=} \int_S \vec{n} \cdot \vec{\nabla} \left(\frac{1}{r}\right) \mathrm{d}S$$

$$\mathrm{d}\ S = r^2\ \mathrm{d}\ \Omega\ ;$$
$$\mathrm{d}\ \Omega = \text{space angle} = \sin\ \theta\ \mathrm{d}\ \phi\ \mathrm{d}\ \theta\ ;$$
$$V = \frac{4\ \pi\ r^3}{3}\ ;$$
$$\int_S \vec{n} \cdot \vec{\nabla} \left(\frac{1}{r} \right) \mathrm{d}\ S = \int_S \underbrace{\frac{\partial}{\partial\ r} \left(\frac{1}{r} \right)}_{-\frac{1}{r^2}} r^2\ \mathrm{d}\ \Omega = -\int_S \mathrm{d}\ \Omega = -4\ \pi\ ;$$
$$\int_0^{\pi} \sin\ \theta\ \mathrm{d}\ \theta \int_0^{2\ \pi} \mathrm{d}\ \phi = 4\ \pi\ ;$$
$$\Delta \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\ \pi\ \delta\ (\vec{x} - \vec{x}')\ ;$$
$$\vec{\nabla} \cdot \vec{E}\ (\vec{x}) = -\Delta\ \phi\ (\vec{x}) = -k \int \mathrm{d}^3\ x'\ \varrho\ (\vec{x}')\ \Delta \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = 4\ \pi\ k\ \varrho\ (\vec{x})\ ;$$

Magnetic fields: One never finds isolated magnetic charges or „monopoles“, only dipoles.

Absence of magnetic monopoles:
$$\vec{\nabla} \cdot \vec{B}\ (\vec{x}) = 0\ ;$$
$$\mathrm{div}\ \vec{B}\ (\vec{x}) = 0\ ;$$

$\mathrm{div}\ \mathrm{curl}\ \vec{A} \equiv 0 \Rightarrow \vec{B}\ (\vec{x}) = \vec{\nabla} \times \vec{A}\ (\vec{x})$
$\vec{A}\ (\vec{x})$: vector potential

$\vec{A}\ (\vec{x})$ is not unique, since
$\vec{A}\ (\vec{x}) \rightarrow \vec{A}\ (\vec{x}) + \vec{\nabla}\ \Lambda\ (\vec{x})$

$\vec{B}\ (\vec{x})$ does not change ($\mathrm{curl}\ \mathrm{grad}\ \Lambda \equiv 0$)

Such a transformation is calles **gauge transformation**.

The total electromagnetic force on a charged particle with charge q is:

$\vec{F} = q\ (\vec{E} + \vec{v} \times \vec{B})$ **(Lorentz force)**

This force is conservative, $\int \vec{F}\ \mathrm{d}\ \vec{x} = 0$, since $\vec{v}$ is tangent to trajectory.

Supplement*: What is the difference between experimental and theoretical physics?*

Theory*: Develops mathematical schemes to solve dynamical equations. Focuses on predictions of a system's dynamical behavior, based on postulated fundamental laws.*

Experiment*: Develops reproducible, accurate measurement techniques. Focuses on measurements of a system's dynamical behavior, in order to check the theoretical* *predictions, with the ultimate goal to check and to reduce the number of fundamental laws.*

3. Lagrangian formulation of mechanics

Sometimes a particle is not free to move in Euclidean 3-space but only in a *restricted* region. Such a system is called a **constrained system**. The particle's area of motion, **configuration manifold**, is neither Euclidean nor 3N-dimensional.

$\Rightarrow$ Equations of motion must include information about the forces that give rise to the constraints.

How Newtons' equations can be rewritten such that constraints are taken into account from the outset $\Rightarrow$ **Lagrangian formulation of dynamics**.

Lagrange equations

- same physical content as Newtons' equations
- easier to apply to dynamical systems
- symmetry
- variational principle

3.1 Constraints and configuration manifolds

3.1.1 Constraints and work

The motion is often constrained by some external agents applying forces that are initially unknown (body gliding on a table, a swinging weight of a pendulum).

Suppose one is dealing with a system of N particles and that the constraints are given by **constraint equations**:

$f_j = (\vec{x}_1, \ldots, \vec{x}_N, t) = 0\ ;\ \ j = 1, 2, \ldots, K < 3N$; (1)

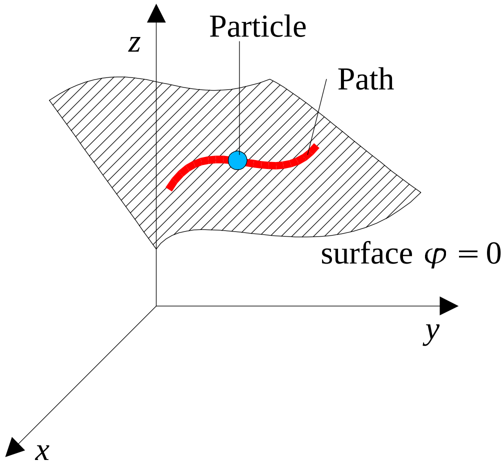
$\vec{x}_i$ are position vectors of the N particles. The f_j are differentiable functions. t -dependence: e.g. surface of a table could be waving.

Constraints of type (1) are calles **holonomic** (means integrable).

More general:
$f_j(\vec{x}_1, \ldots, \vec{x}_N, \dot{\vec{x}}_1, \ldots, \dot{\vec{x}}_N, t) = 0\ ;\ \ j = 1, \ldots, K < 3N$
called **non-holonomic**.

Other:
$f_j(\vec{x}_1, \ldots, \vec{x}_N, t) < 0$

Example: 1 particle $N = 1$; 1 constraint $K = 1$



$m\ddot{\vec{x}} = \vec{F} + \vec{C}$

$\vec{F}(\vec{x}, \dot{\vec{x}}, t)$ known external force

$\vec{C}$ unknown force of constraint that the surface exerts on particle.

$\vec{C}$ force that ensures the particle to stay on $f = 0$ but otherwise to pose no restriction for the particle

$\vec{C} \perp$ surface $f = 0$

$f(\vec{x}, t) = \text{const}$ be the equation of any surface $\Rightarrow \vec{\nabla} f(\vec{x}, t) \perp$ surface provided $\vec{\nabla} f \neq 0$:

$\vec{\nabla} f \neq 0$ on the f=0 surface

Example: $f_a(\vec{x}) = \vec{s} \cdot \vec{x} = 0$

$f_b(\vec{x}) = (\vec{s} \cdot \vec{x})^2 = 0$

$\vec{\nabla} f_a = \vec{s}$ whereas $\vec{\nabla} f_b = 0$

$\Rightarrow$ only $f_a = 0$ is acceptable

If N>1, K>1, the matrix $\left\{ \frac{\partial f_j}{\partial x^\alpha} \right\}$; $\alpha = 1, \dots, 3N$; $j = 1, \dots, K$ be at least of rank K.

$\vec{C} = \lambda \vec{\nabla} f(\vec{x}, t) \qquad \lambda = \lambda(t)$ can be any number.

$$\left. \begin{array}{r} m\ddot{\vec{x}} = \vec{F} + \vec{C} \\ f(\vec{x}, t) = 0 \end{array} \right\} \begin{array}{l} x(t) \\ y(t) \\ z(t) \end{array}$$

unknowns: x(t), y(t), z(t), λ

4 equations for 4 unknowns

$\vec{C}$ is also called **normal force**.

Assume that the external force:

$\vec{f} = -\vec{\nabla} V(\vec{x}, t)$

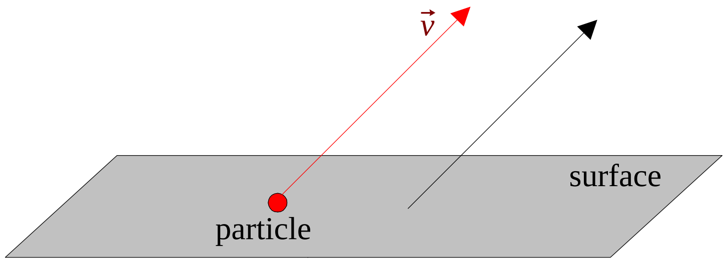
$m\ddot{\vec{x}} \cdot \dot{\vec{x}} \equiv \frac{d}{dt}\left(\frac{1}{2} m \dot{x}^2\right) = -\vec{\nabla} \cdot \dot{\vec{x}} + \lambda \vec{\nabla} f \cdot \dot{\vec{x}}$

$f(\vec{x}(t), t) = 0$

$\frac{df}{dt} = \vec{\nabla} f \cdot \dot{\vec{x}} + \frac{\partial f}{\partial t} = 0$

$\frac{dV}{dt} = \vec{\nabla} V \cdot \dot{\vec{x}} + \frac{\partial V}{\partial t}$

$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2} m \dot{x}^2 + V\right] = \frac{\partial V}{\partial t} - \delta \cdot \frac{\partial f}{\partial t}$



If the surface moves, the particle on it also moves.

Thus: For static surfaces $\left(\frac{\partial f}{\partial t}=0\right)$ constraint forces do no work!

Friction: Here: Only constraint forces that $\propto \vec{\nabla} f$

3.1.2 Generalized coordinates

$$m\ddot{\vec{x}}=\vec{F}+\lambda\vec{\nabla} f$$
$$f(\vec{x}, t)=0$$

Choose: **arbitrary** vector $\vec{\tau}$ tangent to the surface f=0, i.e. $\vec{\tau}\cdot\vec{\nabla} f=0$

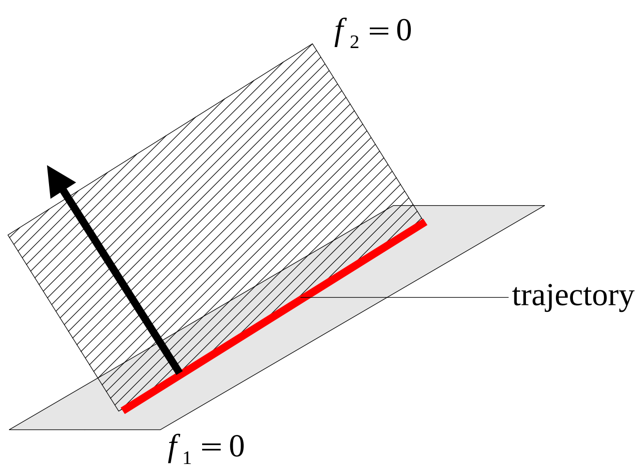
$$\left(m\ddot{\vec{x}}-\vec{F}\right)\cdot\vec{\tau}=0 \quad (*)$$

$\Rightarrow$ 2 linearly independent vectors at each point $\vec{x}$ and hence 2 linearly independent vector functions of $\vec{x}$ and that describe a tangent to the surface.

(*) yields 2 equations for $\vec{x}(t)+f(\vec{x}, t)=0$ ... 3rd equation. $\vec{\tau}\cdot\vec{\nabla} f=0$

1 particle in 3D-space can have at most 2 holonomic constraints.

$$f_1(\vec{x}, t)=0\ ; \quad f_2(\vec{x}, t)=0$$
$$\vec{C}=\lambda_1\vec{\nabla} f_1+\lambda_2\vec{\nabla} f_2$$



System of N particles: K independent holonomic constraints. We drop the summation convention for this section.

$$m_i\ddot{\vec{x}}_i=\vec{F}_i+\vec{C}_i$$

$$\vec{C}_i = \sum_{j=1}^{K} \lambda_j \vec{\nabla}_i f_j$$

$$\frac{\partial V}{\partial t} = o \Rightarrow \frac{dE}{dt} = -\sum_j \lambda_j \frac{\partial f_j}{\partial t}$$

Now let $\vec{\tau}_i$ be N **arbitrary** vectors „tangent to the surface“: $\sum_{i=1}^{N} \vec{\tau}_i \vec{\nabla}_i f_j = 0 \; ; \quad j = 1, \dots, K$

If $\frac{\partial f_j}{\partial x^\alpha}$ is of rank K, this equation gives K independent relations among the 3N components of the N vectors $\vec{\tau}_i$ so that 3N-K of the components $\vec{\tau}_i$ are independent.

$$\sum_i \left(m_i \dot{\vec{x}}_i - \vec{F}_i \right) \cdot \vec{\tau}_i = 0$$

D' Alembert's principle:

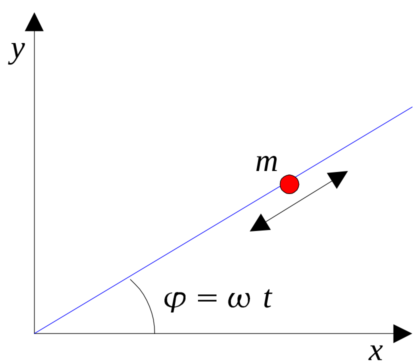
3N – K independent components of $\vec{\tau}_i$.

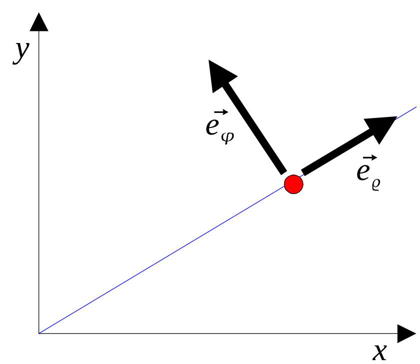
3N – K independent relations. $f_j = 0 \; (j = 1 \dots K)$ provide K other relations.

The dynamical system is constrained to a (3N-K)-dimensional hypersurface: Configuration manifold $\mathbb{Q}$ of the system.

The problem is to pick the 3N-K components of the $\vec{\tau}_i$ that fully characterize the motion in $\mathbb{Q}$

Example:A particle moves along a vigid rod that rotates wih a given angular velocity ω.





$$f_1 = z = 0$$

$$f_2 = \arctan\left(\frac{y}{x}\right) - \omega t = 0$$

$z = \dot{z} = \ddot{z} = 0 \Rightarrow$ eliminate z-coordinates all together

$$\left.\begin{array}{l} x = \rho \cos \varphi \\ y = \rho \sin \varphi \end{array}\right\} \Rightarrow f_2 = \varphi - \omega t = 0$$

The equation of motion is $m \ddot{\vec{x}} = \lambda \, grad \, f_2$

$$grad = \vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi}$$

$$\ddot{\vec{x}} = \left(\ddot{\rho} - \rho \dot{\varphi}^2 \right) \vec{e}_\rho + \left(\rho \ddot{\varphi} + 2 \dot{\rho} \dot{\varphi} \right) \vec{e}_\varphi$$

$$grad \, f_2 = \vec{e}_\varphi \frac{1}{\rho}$$

Component along $\vec{e}_\rho$: $m \ddot{\rho} - m \rho \dot{\varphi}^2 = 0$

Component along $\vec{e}_\varphi$: $m\ \rho\ \ddot{\varphi} + 2\ m\ \dot{\rho}\ \dot{\varphi} = \frac{\lambda}{\rho}$

The projection $\left(m\ \ddot{\vec{x}} - \lambda\ grad\ f_2\right) \cdot \vec{\tau} = m\ \ddot{\vec{x}} \cdot \vec{\tau} = 0 \qquad \left(\vec{\tau} \propto \vec{e}_\rho\right)$

is $m\ \ddot{\rho} - m\ \omega^2\ \rho = 0$ since $\varphi = \omega\ t \Rightarrow \dot{\varphi} = \omega$.

Is a linear homogenous ODE with constant coefficients $\rightarrow \rho\,(t) \propto e^{\kappa\, t}$

$\rho\,(t) = A\ e^{\omega\, t} + B\ e^{-\omega\, t}$

General solution:

(*) $\dot{\varphi} = \omega\ ,\ \ddot{\varphi} = 0$

$z\ m\ \dot{\rho} = \frac{\lambda}{\rho}$

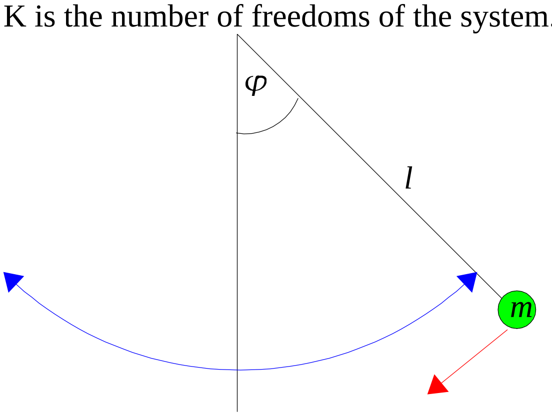
$\vec{C} = \lambda\ grad\ f_2 = 2\ m\ \omega^2\ \vec{e}_\varphi \left(A\ e^{\omega\, t} - B\ e^{-\omega\, t}\right)$

Now we return to our N-particle problem: Picking the $\vec{\tau}_i$ to statisfy:

$$\sum_{i=1}^{N} \vec{\tau}_i\ \vec{\nabla}_i\ f_j = 0\ (j = 1.\ \ldots\ ,K)$$

means picking the generalized tangent vector with 3N-K independent components.

Define 3N-K generalized coordinates $q^{alpha,}$ $\alpha = 1, \ldots, 3N - K$ such that they uniquely determine all particle coordinates $\vec{x}_i = \vec{x}_i\,(q^{1,}\ q^{2,}\ \ldots,\ q^{3N-K},\ t)$ $i = 1, \ldots ,N$ and gurantee the equations of constraint to be obeyed for **any** values of q^α, $f_j\,(\vec{x}_1\,(q^1 \ldots q^{f,}\ t),\ \vec{x}_2\,(q^1 \ldots q^{t,}\ t),\ \ldots\ \vec{x}_N\,(q^1 \ldots q^{f,t})) = 0$ or $f_j\,(x,\,(q^1 \ldots q^{f,}\ t),\ \ldots,\ x_{3N}\,(q^1 \ldots q^{f,}\ t)) = 0$ for arbitrary $\{q^\alpha\}$ and for j= 1,..,K where f = 3N-K is the number of freedoms of the system.



$i = 1\ ,\ \ldots\ ,\ N$

$m_i\ \ddot{\vec{x}}_i = \vec{F}_i + \vec{C}_i\ ;$

$f_j\,(\vec{x}_1\ ,\ \ldots\ ,\ \vec{x}_N\ ,\ t) = 0 \qquad j = 1\ ,\ \ldots\ ,\ K\ ;$

$$\vec{C}_i = \sum_{j=1}^{K} \lambda_j\ \vec{\nabla}_i\ f_j$$

$$\sum_{i=1}^{N} \vec{\tau}_i \cdot \vec{\nabla}_i\ f_j = 0 \qquad j = 1\ ,\ \ldots\ ,\ K$$

$$\sum_{i=1}^{N} (m_i\ \ddot{\vec{x}}_i - \vec{F}_i)\ \vec{\tau}_i = 0$$

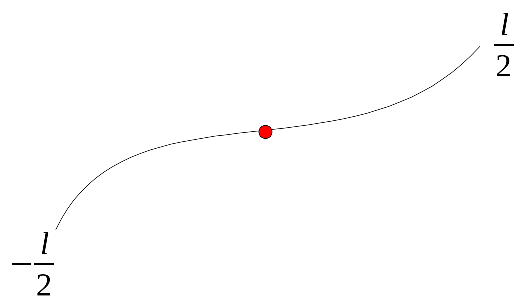
Generalized Coordinates:
$q^{\alpha} : \alpha = 1, \ldots, 3N - K$

- $\vec{x}_i = \vec{x}_i(q^1, q^2, \ldots, q^{3N-K}, t) \quad i = 1, \ldots, N$
- $f_j(\vec{x}_1(q^1, \ldots, q^n, t), \vec{x}_2(q^1, \ldots, q^n, t), \ldots, \vec{x}_N(q^1, \ldots, q^n, t)) = 0$
  $n = 3N - K$ for *any* $\{q^{\alpha}\} \wedge j = 1, \ldots, K$

3.1.3 Examples of configuration manifolds

Greek indices run from 1 to $n = 3N - K$

The finite line

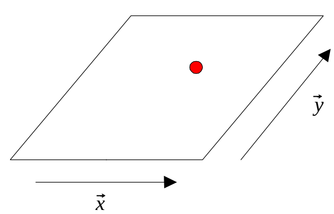


Motion of bead along wire.

$N = 1, K = 2 \Rightarrow n = 1;$
Dimension of $\mathbb{Q} = 1$
Generalized coordinate: $-l/2 \leq q \leq l/2$

The plane

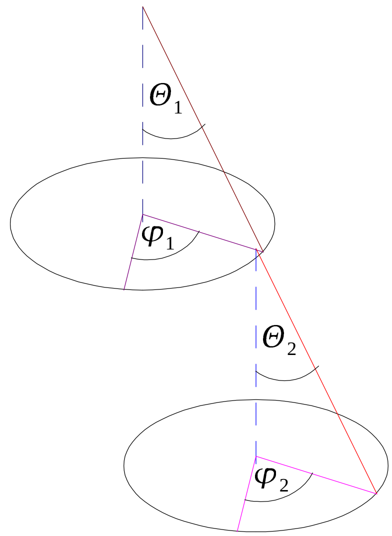
Particle on a table: $N = 1, K = 1 \Rightarrow n = 2;$



$q^1 = x, q^2 = y;$

The Double Pendulum

Spherical pendulum suspended from another are:



$N = 2 , K = 2 ; 3 \cdot 2 - 2 = 4 \Rightarrow \dim \mathbb{Q} = 4$
$q^{\alpha} : \phi_1 , \phi_2 , \theta_1 , \theta_2$

3.2 Lagranges' Equations

3.2.1 Derivation of Lagranges' Equations

$$\sum_{i=1}^{N} (m_i \ddot{\vec{x}}_i - \vec{F}_i) \cdot \vec{\tau}_i = 0$$

$\ddot{\vec{x}}_i$, $\vec{F}_i$ and $\vec{\tau}_i$ use q^{α}

Notation*: We will use the summation convention for greek indices ($1 \ldots n = 3 N - K$), but we continue to use summation signs for $i = 1 \ldots n$.*

Corollary: $\vec{\tau}_i = \varepsilon^{\alpha} \frac{\partial \vec{x}_i (q^1 , \ldots , q^n , t)}{\partial q^{\alpha}}$
where ε^{α} are arbitrary constants ($\alpha = 1 \ldots n$).

Proof: (1) $f_j (q^1 , \ldots , q^n , t) = 0$ must be obeyed for *any* value of $\{ q^{\alpha} \}$.

$$\frac{\partial f_j}{\partial q^{\alpha}} \equiv 0$$

(2) $\sum_{i=1}^{N} \vec{\tau}_i \cdot \vec{\nabla}_i f_j = \varepsilon^{\alpha} \sum_{i=1}^{N} \vec{\nabla}_i f_j \frac{\partial \vec{x}_i}{\partial q^{\alpha}} = \varepsilon^{\alpha} \frac{\partial f_j}{\partial q^{\alpha}} \underset{(1)}{=} 0$.

Choose: $\{ \varepsilon^{\alpha} \} = \{ 1 , 0 , 0 , \ldots \} , \{ 0 , 1 , 0 , \ldots \} , \ldots$:

$$\sum_{i=1}^{N} (m_i \ddot{\vec{x}}_i - \vec{F}_i) \frac{\partial \vec{x}_i}{\partial q^{\alpha}} = 0 \qquad \alpha = 1 , \ldots , n ;$$

$$\vec{F}_i = - \nabla V (\vec{x}_1 , \ldots , \vec{x}_N) ;$$

$$\sum_{i=1}^{N} \vec{F}_i \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\sum_{i=1}^{N} \vec{\nabla}_i V \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\frac{\partial V}{\partial q^\alpha};$$

$$\ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = \frac{d}{dt}\left[\dot{\vec{x}}_i \frac{\partial \vec{x}_i}{\partial q^\alpha}\right] - \dot{\vec{x}}_i \frac{d}{dt}\left(\frac{\partial \vec{x}_i}{\partial q^\alpha}\right);$$

$$\vec{v}_i \equiv \dot{\vec{x}}_i = \frac{d\vec{x}_i}{dt} = \frac{\partial \vec{x}_i}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial \vec{x}_i}{\partial t};$$

$$\frac{\partial \vec{v}_i}{\partial \dot{q}^\alpha} \equiv \frac{\partial \dot{\vec{x}}_i}{\partial \dot{q}^\alpha} = \frac{\partial \vec{x}_i}{\partial q^\alpha};$$

$$\frac{d}{dt}\left(\frac{\partial \vec{x}_i}{\partial q^\alpha}\right) = \frac{\partial^2 \vec{x}_i}{\partial q^\alpha \partial q^\beta} \cdot \dot{q}^\beta + \frac{\partial}{\partial t}\left(\frac{\partial \vec{x}_i}{\partial q^\alpha}\right) = \frac{\partial}{\partial q^\alpha}\left(\frac{\partial \vec{x}_i}{\partial q^\beta} \dot{q}^\beta + \frac{\partial \vec{x}_i}{\partial t}\right) = \frac{\partial \vec{v}_i}{\partial q^\alpha};$$

$$\sum_{i=1}^{N} m_i \ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial \dot{q}^\alpha}\right) - m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q^\alpha}\right] = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha}$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2;$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} + \frac{\partial V}{\partial q^\alpha} = 0 \qquad \alpha = 1, \dots, n$$

$$\frac{\partial V}{\partial \dot{q}^\alpha} = 0;$$

$$(V = V(q^\alpha))$$

$$L = T - V:$$

Lagrangian function or simply Lagrangian $L(q^\alpha, \dot{q}^\alpha, t)$:

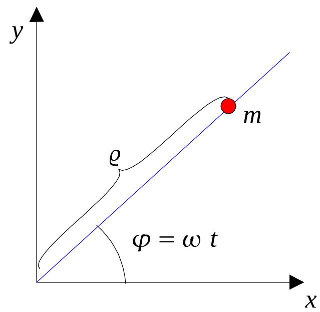
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0 \qquad \alpha = 1, \dots, n = 3N - K;$$

Like Newtons' equations, Lagranges' equations are a set of 2nd-order equations, but now for the $q^\alpha(t)$.

$$\frac{\partial^2 L}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} \ddot{q}^\beta + \frac{\partial^2 L}{\partial q^\beta \partial \dot{q}^\alpha} \dot{q}^\beta + \frac{\partial^2 L}{\partial t \partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0;$$

3.2.2 Examples

1.:



$$q = \varrho$$
$$x = \varrho \cos \omega t$$
$$y = \varrho \sin \omega t$$
$$z = 0$$

$$V = 0$$

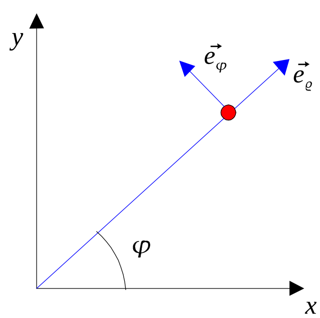
$$L(\varrho, \dot\varrho) = T = \frac{m}{2}(\dot x^2 + \dot y^2 + \dot z^2);$$
$$\dot x = \dot\varrho \cos \omega t - \varrho \omega \sin \omega t;$$
$$\dot y = \dot\varrho \sin \omega t + \varrho \omega \cos \omega t;$$
$$L(\varrho, \dot\varrho) = \frac{m}{2}(\dot\varrho^2 + \omega^2 \varrho^2);$$

$$\frac{\partial L}{\partial \dot\varrho} = m \dot\varrho, \quad \frac{\partial L}{\partial \varrho} = m \omega^2 \varrho;$$

$$\ddot\varrho - \omega^2 \varrho = 0;$$

Old way:

$$f_1 = z = 0;$$
$$f_2 = \underbrace{\arctan\left(\frac{y}{x}\right)}_{\varphi} - \omega t = 0$$

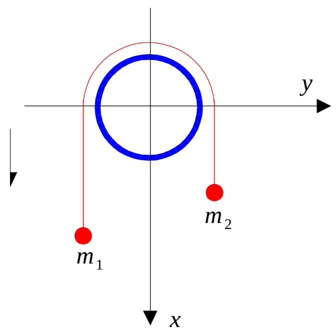


$$m \ddot{\vec x} = \underbrace{\lambda \operatorname{grad} f_2}_{=\vec C};$$

$$\vec C = 2 m \omega^2 \vec e_\varphi (A e^{\omega t} - B e^{-\omega t});$$
$$\varrho(t) = A e^{\omega t} + B e^{-\omega t};$$

2.:

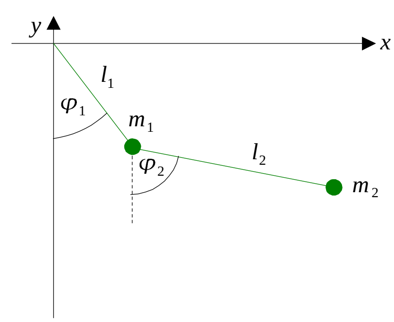
Two particles m_1, m_2 are connected by an inextensible string of negligible mass and of length l which passes over a frictionless pulley of negligible mass.



$q = x_1$ (1 degree of freedom)
$x_1 = q \; , \; y_1 = -R \; , \; z_1 = 0 \; ;$
$x_2 = l - q \; , \; y_2 = R \; , \; z_2 = 0$

$$T = \frac{1}{2} (m \; \dot{x}_1^{\;2} + m_2 \; \dot{x}_2^{\;2}) = \frac{1}{2} (m_1 + m_2) \; \dot{q}^2 \; ;$$
$$V = -m_1 \; g \; x_1 - m_2 \; g \; x_2 = -m_1 \; g \; q - m_2 \; g \; (l - q) = -g \; (m_1 - m_2) \; q + const \; ;$$
$$L \; (q \; , \; \dot{q}) = \frac{1}{2} (m_1 + m_2) \; \dot{q}^2 + q \; (m_1 - m_2) \; q \quad \Rightarrow \quad (m_1 + m_2) \; \ddot{q} = (m_1 - m_2) \; g \; ;$$

3. Planar Double Pendulum:



$x_1 = l_1 \sin \varphi_1$
$y_1 = -l_1 \cos \varphi_1$
$z_1 = 0 \; ;$
$x_2 = l_1 \sin \varphi_1 + l_2 \sin \varphi_2 \; ;$
$y_2 = -l_1 \cos \varphi_1 - l_2 \cos \varphi_2$
$z_2 = 0$

$$T = \frac{m_1}{2} (\dot{x}_1^{\;2} + \dot{y}_1^{\;2}) + \frac{m_2}{2} (\dot{x}_2^{\;2} + \dot{y}_2^{\;2}) = \frac{m_1}{2} \; l_1^2 \; \dot{\varphi}_1^{\;2} + \frac{m_2}{2} \left(l_1^2 \; \dot{\varphi}_1^{\;2} + l_2^2 \; \dot{\varphi}_2^{\;2} + 2 \; l_1 \; l_2 \cos (\varphi_1 - \varphi_2) \; \dot{\varphi} \right.$$
;

$$V = m_1 \; g \;\; y_1 + m_2 \; g \;\; y_2 = -(m_1 + m_2) \; g \; l_1 \cos \varphi_1 - m_2 \; g \; l_2 \cos \varphi_2 \; ;$$

$L = T - V$;

3.2.3 Transformations and Conservation of Energy

Theorem: Let L_1 and L_2 be two Lagrangian functions such that the equations of motion obtained from them are *exactly* the same. Then there exists a function ϕ on the configuration manifold $\mathbb{Q}$ such that

$$L_1 - L_2 = \frac{\mathrm{d}\,\phi}{\mathrm{d}\,t}\ .$$

Proof: Jose and Saletan.

$$\frac{\partial^2 L}{\partial \dot{q}^\beta\, \partial \dot{q}^\alpha}\, \ddot{q}^\beta = G_\alpha(q, \dot{q}, t)$$

Take $q(t_0), \dot{q}(t_0)$: Hessian Condition

$$\det\left(\underbrace{\frac{\partial^2 L}{\partial \dot{q}^\alpha\, \partial \dot{q}^\beta}}_{\text{Hessian Matrix}}\right) \neq 0$$

Problem: Given $L \Rightarrow E = T + V$?

Single particle, cartesian coordinates:

$$\frac{\partial V}{\partial \dot{x}} = 0$$

$$T = \frac{1}{2} m\, \dot{x}^2 \Rightarrow \dot{x}\, \frac{\partial L}{\partial \dot{x}} - L = \underbrace{\dot{x}\, \frac{\partial T}{\partial \dot{x}}}_{2\,T} - T + V = T + V = E$$

General case:

$$E(q, \dot{q}) \equiv \dot{q}^\alpha\, \frac{\partial L}{\partial \dot{q}^\alpha} - L$$

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left[\dot{q}^\alpha\, \frac{\partial L}{\partial \dot{q}^\alpha} - L\right] = \ddot{q}^\alpha\, \frac{\partial L}{\partial \dot{q}^\alpha} + \dot{q}^\alpha\, \underbrace{\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right)}_{\frac{\partial L}{\partial q^\alpha}} - \ddot{q}^\alpha\, \frac{\partial L}{\partial \dot{q}^\alpha} - \dot{q}^\alpha\, \frac{\partial L}{\partial q^\alpha} - \frac{\partial L}{\partial t};$$

$$\frac{\mathrm{d}\,E}{\mathrm{d}\,t} = -\frac{\partial L}{\partial t}$$

or

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\mathrm{d}\,E}{\mathrm{d}\,t} = 0;$$

If $L = T - V$ is time-independent and if $\partial V / \partial \dot{q} = 0 \ \Rightarrow\ E = T + V$.

3.2.4 Charged Particle in an Electromagnetic Field

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$$

In this subsection, use W for energy.

$$F_\alpha = e(E_\alpha + \varepsilon_{\alpha\beta\gamma}\, v^\beta\, B_\gamma)$$

$$\varepsilon_{\alpha\beta\gamma} = \begin{cases} 1 & (\alpha\ \beta\ \gamma) \text{ is a even or cyclic permutation of (1 2 3)} \\ -1 & (\alpha\ \beta\ \gamma) \text{ is anticyclic or odd permutation of (1 2 3)} \\ 0 & \text{otherwise} \end{cases}$$

Levi-Civita antisymmetric tensor density

$\varepsilon_{1\,2\,3} = 1$, $\varepsilon_{2\,3\,1} = 1$, $\varepsilon_{3\,2\,1} = -1$;

$E_\alpha = -\partial_\alpha \varphi - \partial_t A_\alpha$;

$$(\vec{E} = -\vec{\nabla}\varphi - \underbrace{\frac{\partial \vec{A}}{\partial t}}_{\text{Law of induction}});$$

$B_\alpha = \varepsilon_{\alpha\beta\gamma} \partial_\beta A_\gamma$

$(\vec{B} = \operatorname{curl} \vec{A})$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}, \; \partial_t \equiv \frac{\partial}{\partial t}$$

$$m\, \ddot{x}^\alpha = -e\, \partial_\alpha \varphi - e\, \partial_t A_\alpha + e\, \varepsilon_{\alpha\beta\gamma}\, \dot{x}^\beta\, \varepsilon_{\gamma\mu\nu}\, \partial_\mu A_\nu$$

$$v^\alpha \equiv \dot{x}^\alpha$$

$$m\, \dot{v}^\alpha = -e\, \partial_\alpha \varphi - e\, \partial_t A_\alpha + e\, \varepsilon_{\alpha\beta\gamma}\, v^\beta\, \varepsilon_{\gamma\mu\nu}\, \partial_\mu A_\nu$$

$$\varepsilon_{\alpha\beta\gamma}\, \varepsilon_{\gamma\mu\nu} = \delta_{\alpha\mu}\, \delta_{\beta\nu} - \delta_{\alpha\nu}\, \delta_{\beta\mu}$$

$$\left(\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases} \right)$$

$$m\, \dot{v}^\alpha = -e\, \partial_\alpha \varphi - e\, \partial_t A_\alpha + e\, (v^\nu\, \partial_\alpha A_\nu - v^\mu\, \partial_\mu A_\alpha);$$

($\partial_t A_\alpha$ und $\partial_\mu A_\alpha$: $-\dfrac{\mathrm{d}\, A_\alpha(\vec{x}, t)}{\mathrm{d}\, t}$

$$\frac{\mathrm{d}}{\mathrm{d}\, t} = (m\, v^\alpha + e\, A_\alpha) + \partial_\alpha (e\, \varphi - e\, v^\nu\, A_\nu) = 0$$

φ does not depend on v^α, A_α does not depend on v^α $\Rightarrow$

$$L = \frac{1}{2} m\, \dot{x}^2 - e\, \varphi(\vec{x}, t) + e\, \dot{x}^\beta A_\beta(\vec{x}, t) = \frac{1}{2} m\, \dot{x}^2 + e\, \dot{\vec{x}} \cdot \vec{A}(\vec{x}, t) - e\, \varphi$$

$$W = \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} - L$$

$$\Rightarrow W = \frac{1}{2} m\, \dot{x}^2 + e\, \varphi$$

3.3 Central Force Motion

Consider two bodies (mass points) m_1 and m_2, interact through a force directed along the line joining them and depends only on distance between the mass points:

$m_1\, \ddot{\vec{x}}_1 = \vec{F}(\vec{x})$

$m_2\, \ddot{\vec{x}}_2 = -\vec{F}(\vec{x})$

$\vec{F}$ = force that particle 2 exerts on particle 1 and $\vec{x} = \vec{x}_1 - \vec{x}_2$.

$$\mu \, \ddot{\vec{x}} = \vec{F}$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

or

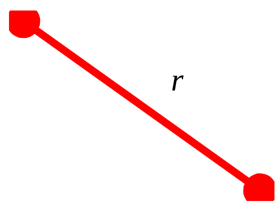
$$\mu = \frac{m_1 \, m_2}{m_1 + m_2}$$

is **reduced mass**.

The **relative motion** of the two bodies single equation of motion:

$\vec{F} = -\vec{\nabla} \, V$;

$V = V \left(|\vec{x}| \right) \equiv V \left(r \right)$



Angular momentum vector $\vec{J} = \mu \, \vec{x} \times \dot{\vec{x}}$ is constant. Since $\vec{J}$ is always perpendicular to both $\vec{x}$ and $\dot{\vec{x}}$, $\vec{x}$ lies always in a fixed plane perpendicular to $\vec{J}$ which is determined by the initial conditions.

Choose: $\vec{J} \, || \,$ z-axis

$$L = \frac{1}{2} \, \mu \, \dot{x}^2 - V \left(r \right) = \frac{1}{2} \, \mu \left(\dot{r}^2 + r^2 \, \dot{\varphi}^2 \right) - V \left(r \right)$$

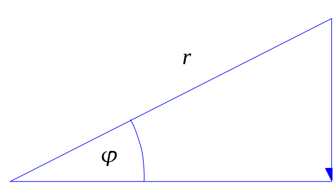
$[\, \dot{\vec{x}} = \dot{\varrho} \, \vec{e}_\varrho + \varrho \, \dot{\varphi} \, \vec{e}_\varphi + \dot{z} \, \vec{e}_z \, ; \;\; \dot{z} = 0 \Rightarrow \dot{x}^2 = \dot{\varrho}^2 + \varrho^2 \, \dot{\varphi}^2 \, ;]$

two freedoms: r , φ

In addition, $L^{total} = L^{relative} + \frac{1}{2} \, M \, \dot{X}^2$

$$\varphi : \frac{\mathrm{d}}{\mathrm{d} \, t} \left(\mu \, r^2 \, \dot{\varphi} \right) = 0$$

$$r : \mu \, \ddot{r} = \mu \, r \, \dot{\varphi}^2 + \frac{\mathrm{d} \, V}{\mathrm{d} \, r} = 0$$



$$v_\perp = r \, \dot{\varphi}$$

$$l = \mu\ r\ v_{\perp} = \mu\ r^2\ \dot{\varphi}$$
$$\mu\ r^2\ \dot{\varphi} = const = l = \text{magnitude of } \left|\vec{J}\right|$$

$$\mu\ \ddot{r} - \frac{l^2}{\mu\ r^3} + \frac{\mathrm{d}\ V}{\mathrm{d}\ r} = 0$$

or

$$\mu\ \ddot{r} - \frac{\mathrm{d}}{\mathrm{d}\ r}\left(\frac{l^2}{2\ \mu\ r^2} + V\ (r)\right) = 0$$

This is equation for a one-dimensional motion of a single particle of mass μ under **effective one-particle potential**.

$$V\ (r) = \frac{l^2}{2\ \mu\ r^2} + V\ (r)$$

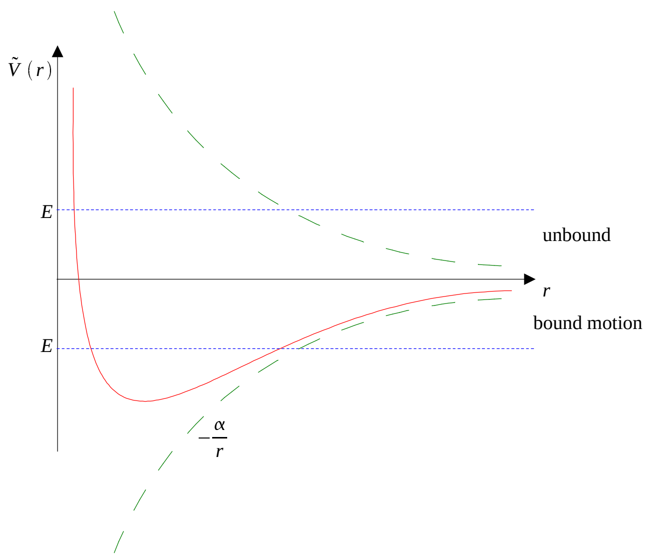
$$\tilde{L}\ (r) = \frac{1}{2}\ \mu\ \dot{r}^2 - \tilde{V}\ (r)$$

$$\tilde{V}\ (r) = \frac{l^2}{2\ \mu\ r^2} + V\ (r)$$

equivalent one-dimensional problem

$V\ (r) = \frac{-\alpha}{r}$ ($\alpha = G\ \mu\ M = G\ m_1\ m_2$ or $\alpha = Z\ e^2$ for $+Z\ e$ (nucleus) and $-e$ (electron))

centrifugal barrier



$$E = T + \tilde{V} = \frac{1}{2}\ \mu\ \dot{r}^2 + \frac{l^2}{2\ \mu\ r^2} + V\ (r) = \frac{1}{2}\ \mu\ \dot{r}^2 + \frac{1}{2}\ \mu\ r^2\ \dot{\varphi}^2 + V\ (r) = const\ ;$$

$$E_{orig} = E + \frac{1}{2}\ M\ \dot{X}^2\ ;$$

($\dot{X} = const$, initial conditions)

3.3.1 The Kepler Problem

Kepler: born in Weil der Stadt

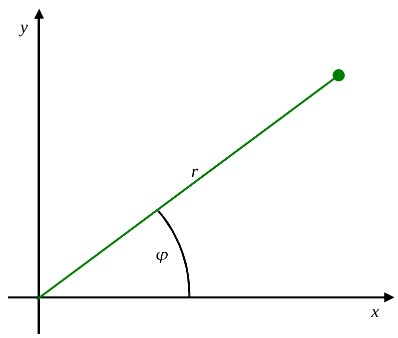
$$V(r) = -\frac{\alpha}{r}, \quad \alpha = G \mu M > 0$$

(K1) the orbit of each planet is an ellipse with the sun at one of is foci

(K2) the position vector from the sun to the planet sweeps out equal areas in equal times

(K3) the period T of each planet's orbit is related to its semimajor axis R so that T^2/R^3 is the same for all planets

$$\mu \ddot{r} - \frac{l^2}{\mu r^3} + \frac{\mathrm{d} V}{\mathrm{d} r} = 0, \; V = -\frac{\alpha}{r}, \; \alpha > 0;$$



$$r = r(t), \; \varphi = \varphi(t) \to r(\varphi(t))$$

$$\frac{\mathrm{d}}{\mathrm{d} t} = \dot{\varphi} \frac{\mathrm{d}}{\mathrm{d} \varphi} = \frac{l}{\mu r^2} \frac{\mathrm{d}}{\mathrm{d} \varphi};$$

$$\frac{\mathrm{d}^2}{\mathrm{d} t^2} = \frac{l^2}{\mu^2 r^2} \frac{\mathrm{d}}{\mathrm{d} \varphi} \left(\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d} \varphi} \right)$$

$$u = \frac{1}{r};$$

$$\frac{\mathrm{d} V}{\mathrm{d} r} \equiv \frac{\alpha}{r^2} = \alpha u^2$$

$$\frac{\mathrm{d}^2}{\mathrm{d} t^2} \left(\frac{1}{u} \right) + \ldots$$

$$\mu \frac{l^2 u^2}{\mu^2} \frac{\mathrm{d}}{\mathrm{d} \varphi} \left(u^2 \underbrace{\frac{\mathrm{d}}{\mathrm{d} \varphi} \frac{1}{u}}_{= -\frac{1}{u^2} \frac{\mathrm{d} u}{\mathrm{d} \varphi}} \right) - \frac{l^2}{\mu} u^3 + \alpha u^2 = 0$$

multiply by $\frac{\mu}{l^2 u^2}$:

$$-\frac{d^2 u}{d\varphi^2} - u + \frac{\alpha\ \mu}{l^2} = 0\ ;$$

$$\frac{d^2 u}{d\varphi^2} + u = \frac{\alpha\ \mu}{l^2}$$

ODE of harmonic oscillator

$$u \equiv \frac{1}{r} = \frac{\alpha\ \mu}{l^2}\left[\varepsilon\ \cos\left(\varphi - \varphi_0\right) + 1\right]$$

ε (called the **eccentricity**) and φ_0 are constants of integration

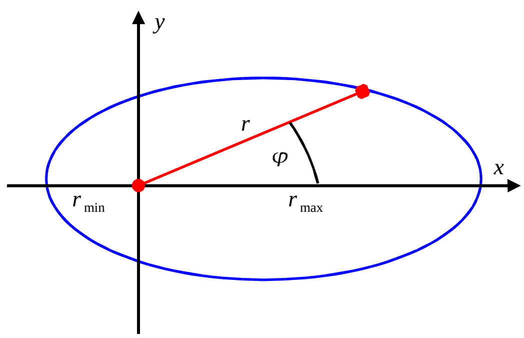
If $\varepsilon = 0 \Rightarrow$ orbit is a closed circle, $\varphi_0 = \pi$

$$\frac{1}{r} = \frac{\alpha\ \mu}{l^2}\left[1 - \varepsilon\ \cos\varphi\right]$$

$0 < \varepsilon < 1$

$\varphi = \pi :\ r\left(\pi\right) = r_{min} = \dfrac{l^2}{\alpha\ \mu\ \left(1 + \varepsilon\right)}$ **perihelion**

$\varphi = 0 :\ r\left(0\right) = r_{max} = \dfrac{l^2}{\alpha\ \mu\ \left(1 - \varepsilon\right)}$ **aphelion**



$$\frac{\left(x - a\ \varepsilon\right)^2}{a^2} + \frac{y^2}{b^2} = 1\ ;\ a = \frac{p}{1 - \varepsilon^2}\ ,\ b = \frac{p}{\sqrt{1 - \varepsilon^2}}\ ,\ p = \frac{l^2}{\alpha\ \mu}\ ;$$

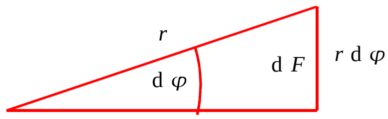
$r = r_{min} :\ v \perp$ position vector to the sun $\Rightarrow\ l = r_{min}\ \mu\ v$

$$T = \frac{1}{2}\ \mu\ v^2 = \frac{1}{2}\ \mu\ \frac{l^2}{r_{min}^2\ \mu^2} = \frac{\mu\ \alpha^2\ \left(\varepsilon + 1\right)^2}{2\ l^2}\ ;$$

$$V = -\frac{\alpha}{r_{min}}\ ,\ E = T + V = \frac{\mu\ \alpha^2\ \left(\varepsilon + 1\right)^2}{2\ l} - \frac{\alpha^2\ \mu\ \left(1 + \varepsilon\right)2}{l^2 \cdot 2} = \frac{\mu\ \alpha^2\ \left(\varepsilon^2 - 1\right)}{2\ l^2}\ ;\ \varepsilon = \sqrt{1 + \frac{2\ E\ l^2}{\mu\ \alpha^2}}\ ;$$

If $\varepsilon < 1$ ($E < 0$) $\Rightarrow$ closed orbit
If $\varepsilon \geq 1$ ($E > 0$) $\Rightarrow$ open orbit, hyperbolic ($\varepsilon > 1$) or parabolic ($\varepsilon = 1$)

(K2): $\mu\ r^2\ \dot{\varphi} = l = const$



$$\frac{1}{2} r^2 \, d\varphi = dF ;$$

$$\frac{1}{2} r^2 \dot{\varphi} = \dot{F} = const ;$$

3.4 The Tangent Bundle TQ

Lagranges' equations are 2nd-order ODE on the configuration manifold $\mathbb{Q}$.

Velocity phase manifold $T \mathbb{Q}$.

Consider 2 particles move on the 2-D surface of a sphere: S^2
The velocity vector of each particle is tangent to the sphere:
$\vec{v} \in \{\text{tangent planes of all points} \in S^2\}$

The difference between the velocity vectors (at two different points) does not lie in either tangent plane.

The Lagrangian $L(q, \dot{q}, t)$; $q \equiv \{ q^\alpha | \alpha = 1, \ldots, n = 3N - K \}$:
L depends on a larger manifold that is called $(T \mathbb{Q})$ **velocity phase manifold**.
$\dim(T \mathbb{Q}) = 2n \quad q, \dot{q}$
Tangent bundle or **tangent manifold** of $\mathbb{Q}$.

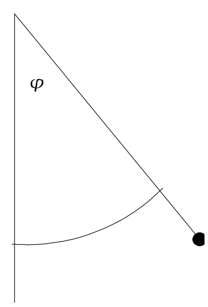
The space $T \mathbb{Q}$ is obtained from $\mathbb{Q}$ by adjoining to each point $q \in \mathbb{Q}$ the vector space, called the tangent space $T_q \mathbb{Q}$, of all possible velocities at q , which are all tangent to $\mathbb{Q}$ at this point.

$T \mathbb{Q}$ is made up of $\mathbb{Q}$ plus all the $T_q \mathbb{Q}$ for $q \in \mathbb{Q}$.

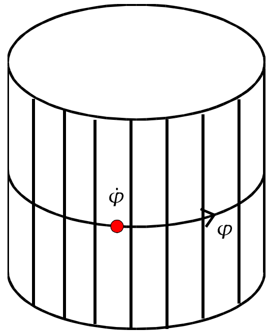
Each point in $T \mathbb{Q}$ is $(q, \dot{q}) = \{(q^\alpha, \dot{q}^\alpha), \alpha = 1, \ldots, n\}$ analogous to (x, v)

Through each point $(q, \dot{q}) \in T \mathbb{Q}$ passes just one solution of the E M, one phase trajectory.
Therefore phase portraits can be constructed in $T \mathbb{Q}$.

Example: Plane pendulum



$$\mathbb{Q} = S' = \{ \varphi | 0 \le \varphi \le 2\pi \} ; \; \dot{\varphi} \in (-\infty, \infty)$$



$T\ S'$ is a cylinder to each $\varphi \in S'$, we attach an infinite line. The line attached to φ is $T_{\varphi}\ S'$ and is called the **fiber above** φ.

On $T\ \mathbb{Q}$, the Lagranges' equations are a set of *first-order ODEs* for the $q^{\alpha}(t)$ and the $\dot{q}^{\alpha}(t)$.

n equations: $H(q, \dot{q}, t)\frac{\mathrm{d}}{\mathrm{d}\ t}(\dot{q}) = G(q, \dot{q}, t)$

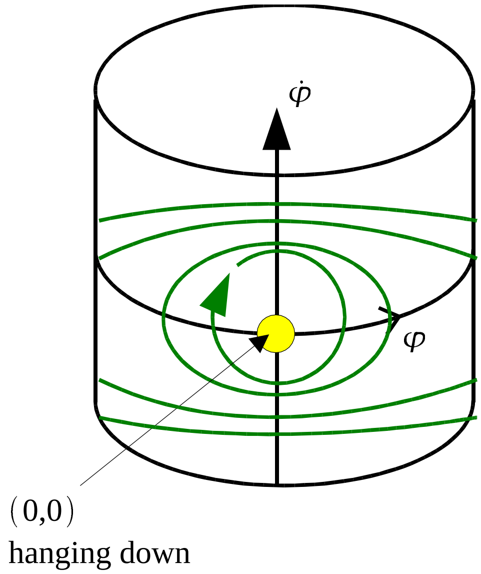
n equations: $\dot{q} = \frac{\mathrm{d}}{\mathrm{d}\ t}(q)$

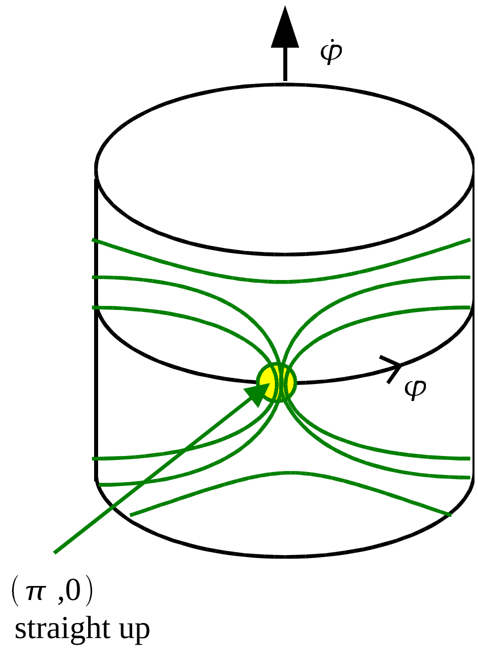
$\Rightarrow$ $2\ n$ equations for $2\ n$ unknowns

The initial conditions for these $2\ n$ 1st-order ODEs are the $2\ n$ initial values $(q_0, \dot{q}_0) \equiv \{q^{\alpha}(0), \dot{q}^{\alpha}(0)\}$.

Thus, given the initial point in $T\ \mathbb{Q}$, the rest of the trajectory is uniquely determined analytically by the $E_0\ M$ or graphically by the phase portraits.

Example: Plane pendulum, $E = \frac{1}{2}\ m\ l^2\ \dot{\varphi}^2 - m\ g\ h\ \cos\varphi = const$


$\dot{\varphi}$
φ
(0,0)
hanging down


$\dot{\varphi}$
φ
(π ,0)
straight up

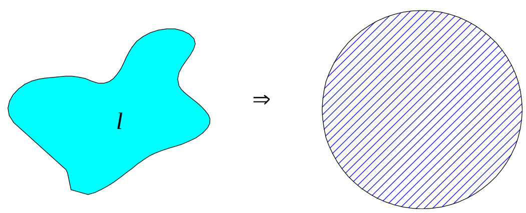
4. Topics in Lagrangian Dynamics

4.1 The Variational Principle and Lagranges' Equations

The Action

Examples of varionational problems:

A. Given a fence of a fixed length. What shape provides the largest area it can surround? Answer: Circle!

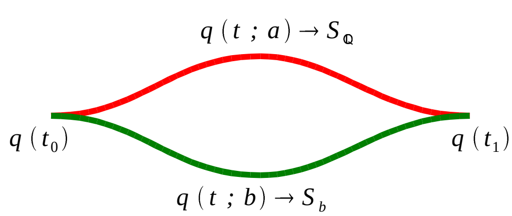


B. Optimization: How to choose some parameters in order to make some function or value an extremum?

Show: a dynamical system moves so as to minimize the action

$$S \equiv \int_{t_0}^{t_1} L(q, \dot{q}, t)\, \mathrm{d}t$$

trajectory $q(t)$: S is an functional of $q(t)$, depends on $q(t)$ and all of $t \in [t_0, t_1]$



$$S[q(t)] \stackrel{!}{=} \text{Minimum}$$, domain = function space

> **Theorem (Hamilton's variational principle)**: The physical trajectory is the one for which S is a minimum.

Families of trajectories starting at $q(t_0)$ and ending at $q(t_1)$: $q(t; \varepsilon)$

$\varepsilon = 0$: $S = \text{Minimum}$

$$\left.\frac{\mathrm{d}S}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{t_0}^{t_1} L(q, \dot{q}, t)\, \mathrm{d}t\right]_{\varepsilon=0} = 0$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \equiv \delta$$

$$\delta S = \delta \int_{t_0}^{t_1} L(q, \dot{q}, t)\, \mathrm{d}t = 0;$$

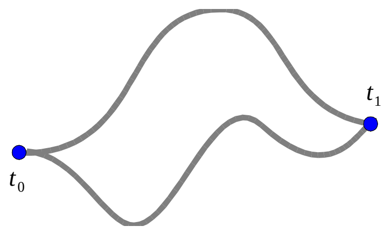
$$\delta L = \frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha ;$$

$$\frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \frac{\mathrm{d}}{\mathrm{d} t} (\delta q^\alpha) = \frac{\mathrm{d}}{\mathrm{d} t} \left[\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right] - \left[\frac{\mathrm{d}}{\mathrm{d} t} \frac{\partial L}{\partial \dot{q}^\alpha} \right] \delta q^\alpha ;$$

Order of $\mathrm{d}/\mathrm{d} t$ and $\mathrm{d}/\mathrm{d} \varepsilon$ is arbitrary.

$$\delta L = \left[\frac{\partial L}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d} t} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) \right] \delta q^\alpha + \frac{\mathrm{d}}{\mathrm{d} t} \left[\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right]$$

$$0 = \delta S = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d} t} \left(\frac{\partial L}{\delta \dot{q}^\alpha} \right) \right] \delta q^\alpha \mathrm{d} t + \underbrace{\int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d} t} \left[\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right] \mathrm{d} t}_{\left. \frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha \right|_{t_0}^{t_1}} = 0$$



$$\int_{t_0}^{t_1} [\ldots] \delta q^\alpha \mathrm{d} t = 0$$

applies to *any* ε -family of trajectories
δq^α are arbitrary (vanish at end points)

$$\Rightarrow \quad \frac{\partial L}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d} t} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) = 0 .$$

$$\int_{t_0}^{t_1} f_\alpha h_\alpha \mathrm{d} t = (f , h)$$

$f , h \in F$ vector space with components $\{ f_\alpha \} , \{ h_\alpha \}$

If $(\Lambda , \delta q) = 0$ for arbitrary $\delta q \in F \;\; \Rightarrow \Lambda = 0 \, (\Lambda \perp \delta q)$

$$\Lambda_\alpha = \frac{\partial L}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d} t} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right)$$

Inclusion of constants:

Given: $\mathbb{Q} , L \in T \mathbb{Q}$; $\dim (T \mathbb{Q}) = 2 n$
Wish to add additional constraints, may be holonomic or nonholonomic
$f_j (q , \dot{q} , t) = 0 \qquad j = 1 , \ldots , K$ (*)
where $K < n$ are the additional constraints.

Variational method: We now require that the **comparison paths** satisfy the new constraints.

Now: $\delta q \in F \left(\delta q_\alpha \equiv \frac{\partial q^\alpha}{\partial \varepsilon} \right)$ are not arbitrary but are restricted by (*)

$$\frac{\partial f_j}{\partial \varepsilon} \equiv \frac{\partial f_j}{\partial q^\alpha} \frac{\partial q^\alpha}{\partial \varepsilon} + \frac{\partial f_j}{\partial \dot{q}^\alpha} \frac{\partial \dot{q}^\alpha}{\partial \varepsilon} = 0$$

Multiply each of these K equations by an arbitrary sufficiently well behaved function $\lambda_j(t)$:

$$\int_{t_0}^{t_1} \sum_{j=1}^{K} \left[\lambda_j \frac{\partial f_j}{\partial q^\alpha} \frac{\partial q^\alpha}{\partial \varepsilon} + \lambda_j \frac{\partial f_j}{\partial \dot{q}^\alpha} \frac{\partial \dot{q}^\alpha}{\partial \varepsilon} \right] \mathrm{d}\, t = 0$$

Use same trick as before to convert

$$\frac{\partial f}{\partial \dot{q}^\alpha} \frac{\mathrm{d}}{\mathrm{d}\, t} \left[\frac{\partial q^\alpha}{\partial \varepsilon} \right] \to \frac{\mathrm{d}}{\mathrm{d}\, t} (\ \ ) - \ldots \delta q^\alpha$$

$$\int_{t_0}^{t_1} \sum_j \left[\lambda_j \frac{\partial f_j}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\, t} \left(\lambda_j \frac{\partial f_j}{\partial \dot{q}^\alpha} \right) \right] \frac{\partial q^\alpha}{\partial \varepsilon} \mathrm{d}\, t = \left(\sum_j \Lambda_j , \delta q \right) = 0 ;$$

„allowed“ δq -vectors that satisfy $f_j = 0$.

Note that we still have $(\Lambda , \delta q) = 0$, $\Lambda \perp \delta q$, $\delta q \perp \sum_j \Lambda_j \ \Rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \frac{\partial}{\partial \dot{q}^\alpha} \left(L + \sum_{j=1}^{K} \lambda_j f_j \right) - \frac{\partial}{\partial q^\alpha} \left(L + \sum_{j=1}^{K} \lambda_j f_j \right) = 0$$

λ_j are called **Lagrange multipliers**

$\tilde{L} = L + \sum_j \lambda_j f_j$

If constraints are *holonomic*:

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} - \sum_j \lambda_j \frac{\partial f_j}{\partial q^\alpha} = 0 \qquad \alpha = 1 , \ldots , n$$

$$f_j(q , t) = 0 \qquad j = 1 , \ldots , K$$

$$\Rightarrow (q^\alpha , \lambda_j)$$

For *nonholonomic constraints*:

If constraints depend linearly on general velocities, use a trick:

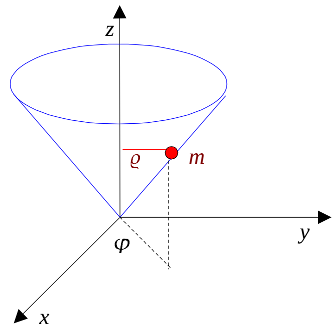
Assume $f_j = f_j(q) = 0$ are holonomic

$\Rightarrow$ then we define *new constraints*

$$g_j \equiv \frac{\mathrm{d}\, f_j}{\mathrm{d}\, t} = \frac{\partial f_j}{\partial q^\alpha} \dot{q}^\alpha = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} - \sum_j \lambda_j \frac{\partial g_j}{\partial \dot{q}^\alpha} = 0$$

Example: A particle of mass m that moves under the influence of gravity on the inner surface of a cone $\sqrt{x^2 + y^2} = a\, z$ which is assumed to be frictionless.



$V = m\ g\ z$

$L = \frac{1}{2}\ m\ (\dot{\varrho}^2 + \varrho^2\ \dot{\varphi}^2 + \dot{z}^2) - m\ g\ z$

Additional constraints: $f = \varrho - a\ z = 0 \quad (K = 1)$

$(\varrho\ ,\ \varphi\ ,\ z) \to \frac{\partial\ f}{\partial\ \varrho}\ ,\ \frac{\partial\ f}{\partial\ \varphi}\ ,\ \frac{\partial\ f}{\partial\ z}$

(1) $\frac{\mathrm{d}}{\mathrm{d}\ t}\left(\frac{\partial\ L}{\partial\ \dot{\varrho}}\right) - \frac{\partial\ L}{\partial\ \varrho} = \lambda \quad \Rightarrow \quad m\ (\ddot{\varrho} - \varrho\ \dot{\varphi}^2) = \lambda$

(2) $\frac{\mathrm{d}}{\mathrm{d}\ t}\left(\frac{\partial\ L}{\partial\ \dot{\varphi}}\right) - \frac{\partial\ L}{\partial\ \varphi} = 0 \quad \Rightarrow \quad m\ \frac{\mathrm{d}}{\mathrm{d}\ t}\ (\varrho^2\ \dot{\varphi}) = 0 \quad \Rightarrow \quad m\ \varrho^2\ \dot{\varphi} = l$

(3) $\frac{\mathrm{d}}{\mathrm{d}\ t}\left(\frac{\partial\ L}{\partial\ \dot{z}}\right) - \frac{\partial\ L}{\partial\ z} = -\lambda\ a \quad \Rightarrow \quad m\ \ddot{z} = -m\ g - \lambda\ a$

Inserting (2) into (1) and setting $a\ \ddot{z} = \ddot{\varrho}$

$\Rightarrow \quad \lambda\ (t) = \frac{1}{1 + a^2}\left(-m\ g\ a - \frac{l^2}{m\ \varrho^3}\right)$

$\ddot{\varrho} = \frac{1}{2}\left(\frac{l}{m}\right)^2 \frac{1}{\varrho^3} - \frac{g}{2}$

$\dot{\phi} = \frac{l}{m\ \varrho^2}$

$z = \varrho$

4.2 Molecular dynamics

$V\ (r_{ij}) = 4\ \varepsilon\left[\left(\frac{\sigma}{r_{ij}}\right)^{12} - \left(\frac{\sigma}{r_{ij}}\right)^{-6}\right] \quad r_{ij} \le r_c$

first fraction: resistance to compression, repulsion at close range provided by closed inner shells of atoms
second fraction: attraction leading to binding

Liquid inert gas (argon):
Van-der-Waals attraction

$r_{ij} = \left|\vec{r}_i - \vec{r}_j\right|$, ε governs the strength of interaction, σ length scale

$$\vec{F}_{ij} = \left(\frac{48\ \varepsilon}{\sigma^2}\right)\left[\left(\frac{\sigma}{r_{ij}}\right)^{14} - \frac{1}{2}\left(\frac{\sigma}{r_{ij}}\right)^{8}\right]\vec{r}_{ij} \Rightarrow$$

$$m\ \ddot{\vec{r}}_i = \sum_{\substack{j=1\\(j\neq i)}} \vec{F}_{ij} \equiv \vec{F}_i$$

Kinetic energy per atom is

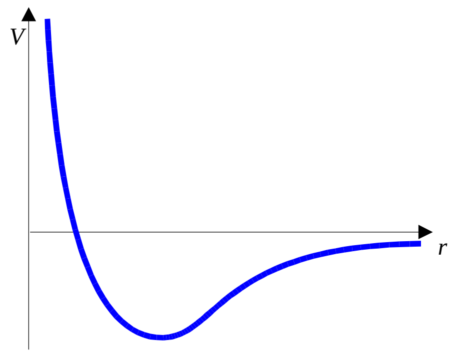
$$\frac{K}{N} = \frac{m}{2\ N}\sum_{i=1}^{N} v_i^2$$

Macroscopic temperature of an ensemble of particles:

$$\frac{3}{2}\ k_B\ T = \frac{m}{2\ N}\sum_{i=1}^{N} v_i^2$$

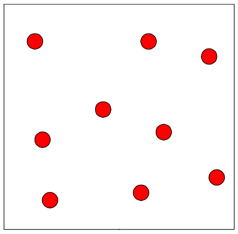
$k_B = 1.38065 \cdot 10^{-23}\ \mathrm{J\ K^{-1}}$ **Boltzmann constant**

Lennard-Jones, Los Alamos National Laboratories



Boundary Conditions:

Homogenous bulk systems: solids, liquids, gases in container. $N \sim 10^{29}\ \mathrm{m}^{-3}$, Computer: $N < 10^7$ atoms .



Number of atoms near walls:

$$\sim N^{\frac{2}{3}} \sim 10^{19}\ \frac{1}{\mathrm{m}^2}$$

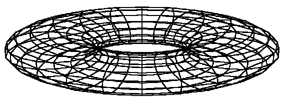
$\Rightarrow 1 : 10^{10}$ atoms

Simulation: $N \sim 1000$: roughly $N \sim 500$ lie close to walls!

Periodic boundary condition

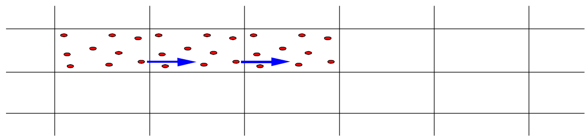
1-D:

2-D: Torus



>3-D: n-dimensional Torus

Infinite space-filling array of identical copies of the simulation region:



2 consequences:

- Atom leaves to the right $\Rightarrow$ reenters to the left
- Atoms lying within r_C of a boundary interact with atoms in an adjacent copy of the system $\Rightarrow$ **wraparound effect**

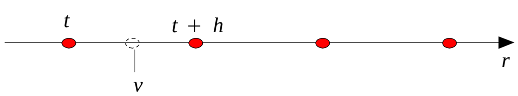
Integration:

- Runge-Kutta Method
- Leapfrog method („Bockspringer“)

$h = \Delta t$ timestep

$$v_{xi}\left(t+\frac{h}{2}\right) = v_{xi}\left(t-\frac{h}{2}\right) + h\ a_{xi}(t);$$

$$r_{xi}(t+h) = r_{xi}(t) + h\ v_{xi}\left(t+\frac{h}{2}\right)$$



Errors:

$O(h^4)$ for r

$O(h^2)$ for v

Typical timesteps for atoms: $1\ \text{fs} = 10^{-15}\ \text{s}$

Constraint method for constant temperature

Initialize: random positions $\vec{r}$, random velocities $\vec{v}$ such that:
$\sum v_i^2 \overset{!}{=}$ given temperature

$$\frac{1}{2} m \sum_{i=1}^{N} \dot{\vec{r}}_i^{\,2} = \alpha T = const$$

as constraint.

$$\ddot{\vec{r}}_i = \frac{1}{m} \vec{F}_i + \lambda \dot{\vec{r}}_i ; \quad \text{multiply by } \dot{\vec{r}}_i$$

$$\dot{T} = 0 \Rightarrow \sum_i \dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i = 0 ;$$

$$\Rightarrow \lambda = - \frac{\sum \dot{\vec{r}}_i \cdot \vec{F}_i}{m \sum_i \dot{r}_i^{\,2}}$$

Modify the leapfrog velolcity equation:

$$\vec{v}_i \left(t + \frac{h}{2} \right) = (1 + \lambda h) \vec{v}_i \left(t - \frac{h}{2} \right) + h \left(1 + \frac{\lambda h}{2} \right) \vec{F}_i (t)$$

where

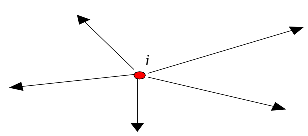
$$\lambda = - \frac{\sum_i \vec{v}_i (t) \cdot \vec{F}_i (t)}{\sum_i v_i^2 (t)}$$

and

$$\vec{v}_i (t) = \vec{v}_i \left(t - \frac{h}{2} \right) + \frac{h}{2} \vec{F}_i (t)$$

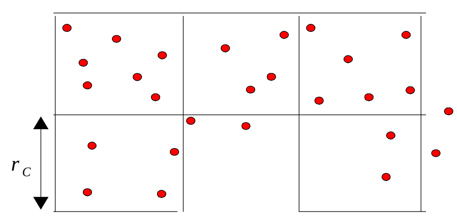
Interaction computations: Cell subdivision

Force felt by 1 particle:



effort $\propto O(N^2)$

Concept of cell subdivision:
Divide simulation region into a lattice of small cells with cell edges slightly exceed r_c in length.



Atoms are assigned to cells on the basis of their current positions $\Rightarrow$ only atoms in adjacent cells interact.

Because of symmetry: only half the neighboring cells need to be considered.

Data organization: linked lists
Economize storage: linked list associates a pointer with each particle that points to the next until a pointer value 0 is encountered, terminating the list. A separate list is required for each cell.

4.3 Cyclic coordinates

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial\,L}{\partial\,\dot{q}^{\alpha}}\right)=\frac{\partial\,L}{\partial\,q^{\alpha}}\;;$$

q^{β} does not appear in L

$$\Rightarrow\;\frac{\partial\,L}{\partial\,q^{\alpha}}=0\Rightarrow\frac{\partial\,L}{\partial\,\dot{q}^{\alpha}}=const\text{ of motion}$$

$$P_{\beta}\,(q\;,\;\dot{q}\;,\;t)\equiv\frac{\partial\,L}{\partial\,\dot{q}^{\beta}}$$

Generalized momentum conjugate to q^{β} :
Such coordinate q^{β} is calles **cyclic** or **ignorable coordinate**.
If initial phase point lies on any submanifold whose equation is of the form $P_{\beta}=C$, the motion stays on that submanifold.

Example 1:

$$L=\frac{1}{2}\,\mu\,(\dot{r}^2+r^2\,\dot{\varphi}^2)-V\,(r)$$

$$\Rightarrow\;P_{\varphi}\equiv\frac{\partial\,L}{\partial\,\dot{\varphi}}=\mu\;r^2\;\dot{\varphi}=\text{conserved}=\tilde{C}$$

Example 2: A free particle in cartesian coordinates

$$L=\frac{1}{2}\,m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$\Rightarrow\;P_x\equiv\frac{\partial\,L}{\partial\,\dot{x}}=m\;\dot{x}=\text{conserved}$$

> ***Note***: *If q^{λ} is ignorable, then $\partial\,L/\partial\,q^{\beta}=0$, which means that L does not change as q^{λ} varies, or* ***L is invariant under translation in the q^{α} direction in $T\,\mathbb{Q}$***.

Example 3:

$$L=\frac{1}{2}\,m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)-m\;g\;z\;.$$

4.4 Dissipative Forces in the Lagrangian Formalism

$$\sum_{i\,0\,1}^{N}(m_i\,\ddot{\vec{x}}_i-\vec{F}_i)\,\frac{\partial\,\vec{x}_i}{\partial\,q^{\alpha}}=0\qquad\alpha=1\;,\;\dots\;,\;n$$

$$\vec{F}_i=-\vec{\nabla}\;V\,(\vec{x}_1\;,\;\dots\;,\;\vec{x}_N)$$

$$\sum_{i=1}^{N}\vec{F}_i\,\frac{\partial\,\vec{x}_i}{\partial\,q^{\alpha}}=-\sum_i\vec{\nabla}_i\;V\;\frac{\partial\,\vec{x}_i}{\partial\,q^{\alpha}}=-\frac{\partial\,V}{\partial\,q^{\alpha}}$$

Now: $\vec{F}_i = \vec{F}_{P_i} + \vec{F}_{D_i}$
where $\mathrm{curl}_i\, \vec{F}_{P_i} = 0$ but $\mathrm{curl}_i\, \vec{F}_{D_i} \neq 0$

$$\sum_i \vec{F}_{D_i} \frac{\partial \vec{x}_i}{\partial q^\alpha} \equiv D_\alpha$$
$$\sum_i \vec{F}_i \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\frac{\partial V}{\partial q^\alpha} + D_\alpha$$
$$\frac{\mathrm{d}}{\mathrm{d}\, t} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} = D_\alpha$$

$$L = T - V$$
$$\vec{F}_{D_i} = -b_i\, \vec{v}_i \qquad b_i > 0$$

These forces do negative work on the particle as it moves $\Rightarrow$ leads to energy loss. Forces are called **dissipative**.

$$D_\alpha = -\sum b_i\, \vec{v}_i \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\sum b_i\, \vec{v}_i \frac{\partial \vec{v}_i}{\partial \dot{q}^\alpha} = -\frac{\partial \tilde{F}}{\partial \dot{q}^\alpha}\,,\ \tilde{F} = \sum_{i=1}^{N} \frac{1}{2} b_i\, v_i^2$$

***Note**: $\vec{v}_i$ are the* real *velocities, $\dot{q}^\alpha$ are generalized velocities.*
*$\tilde{F}$ is called the **Rayleigh function**.*

The rate of energy loss:
$$\frac{\mathrm{d}\, E}{\mathrm{d}\, t} = \frac{\mathrm{d}}{\mathrm{d}\, t} \left(\dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} - L \right) = \ddot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} + \dot{q}^\alpha \frac{\mathrm{d}}{\mathrm{d}\, t} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\mathrm{d}\, L}{\mathrm{d}\, t}$$
$$= \dot{q}^\alpha \left(\frac{\mathrm{d}}{\mathrm{d}\, t} \left(\frac{\partial L}{\partial \dot{q}^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} \right) = -\dot{q}^\alpha \frac{\partial \tilde{F}}{\partial \dot{q}^\alpha}$$

Cartesian coordinates:
$$-\sum_i \vec{v}_i\, \vec{F}_{D_i} = \text{Power loss} = \text{rate of energy loss}$$

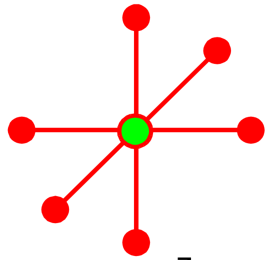
Example: Equilibrian lattice constant of an inert gas crystal

The lattice constant of a simple cubic lattice. Crystal: periodic.
Equilibrium: Atoms at rest
$$E_{tot} = \frac{1}{2} \sum_j V(r_{ij})\,;$$
$$r_{ij} = \left| \vec{r}_i - \vec{r}_j \right|$$

Atoms: denumerable set
$$\vec{r}_i = a\,(n\, \vec{e}_x + m\, \vec{e}_y + k\, \vec{e}_z) = a\, \vec{u}_{nmk} \qquad n, m, k \in \mathbb{Z} \ (\text{integers})$$
a : lattice constant

$$\vec{r}_{ij} : (i, j) \rightarrow (i, \underbrace{i\ j}_{l})$$

$$E_{tot} = \frac{1}{2} \sum_{i\,l} V(r_l) = \frac{1}{2} N \sum_l V(r_l)$$



$$E_{tot} = \frac{1}{2} N \ 4 \varepsilon \left[\sum_{n\,m\,k} \left(\frac{\sigma}{a \ u_{nmk}} \right)^{12} - \sum_{n\,m\,k} \left(\frac{\sigma}{a \ u_{nmk}} \right)^{6} \right]$$

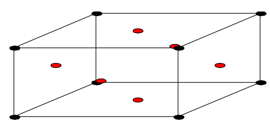
$a < r_c < 2 \ a$

$$\sum_{n\,m\,k} {}' \ u_{nmk}^{-12} = 6 \quad (\sum {}' \text{ means: } n = m = k = 0 \text{ excluded})$$

$$\sum_{n\,m\,k} {}' \ u_{nmk}^{-6} = 6$$

$$\frac{\mathrm{d} \ E_{tot}}{\mathrm{d} \ a} = 0 = -2 \ N \ \varepsilon \left[12 \cdot 6 \ \frac{\sigma^{12}}{a^{13}} - 6 \cdot 6 \ \frac{\sigma^{6}}{a^{7}} \right] \Rightarrow \frac{a}{\sigma} = 2^{\frac{1}{6}} = 1.12$$

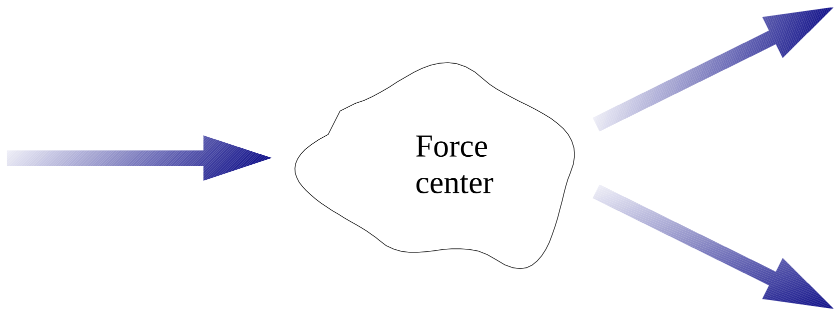
Experimentally: all inert gas crystals have lattice constants where $a / r \approx 1.09$ (face centered cubic)



5. Scattering and Linear Oscillators

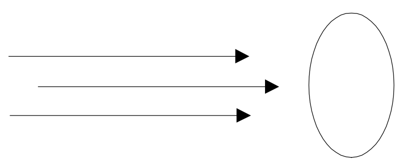
5.1 Scattering

Consider a central-force dynamical system that possesses unbounded orbits.



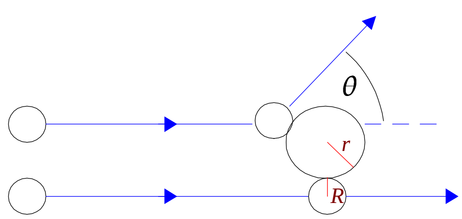
Angle between the distant incoming velocity vector and the distant outgoing one is the **scattering angle**.

Consider beam of noninteracting particles incident on a stationery target consists of a collection of other particles.
The **flux density** or **intensity** I of beam = number of particles crossing a unit area perpendicular to the beam per unit time.

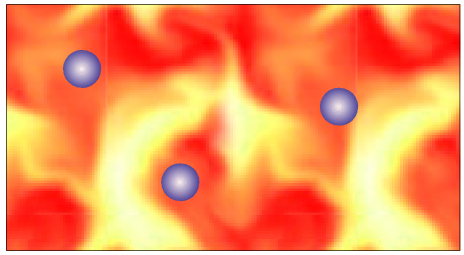


A = beam's cross-sectional area
n = number of target particles per unit area = density of target particles

Assume: beam particles interact with target particles through a hard-sphere interaction
⇒ Cross-sectional area of this interaction is $\sigma = \pi\,(r+R)^2$ where
r = radius of target particle
R = radius of beam particle



⇒ The total interaction cross section that the target particles present to the beam is
$\sum = n\;A\,\sigma - N\;\sigma$
N = number of target particles within the cross-sectional area of the beam
⇒ On the average, the number of collisions per unit time = $I\;\sum$



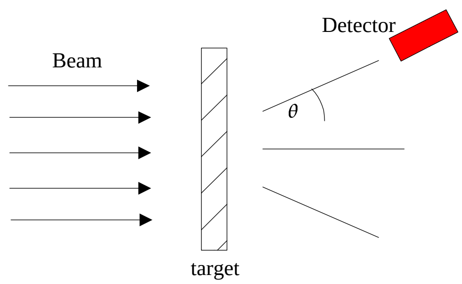
$\sigma = \pi (r+R)^2$

Scattering rate: number of particles scattered out of the beam per unit time =
$S = I \sum = I N \sigma$

For arbitrary interactions: total cross section σ_{tot} per particle:

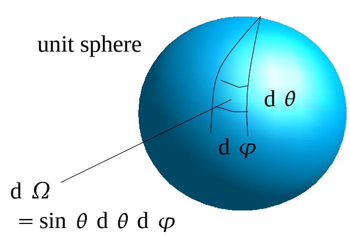
$$\sigma = \frac{\text{out-flux}}{\text{in-flux per unit area}} [\text{area}]$$

In experiment: place a detector at some distance from target



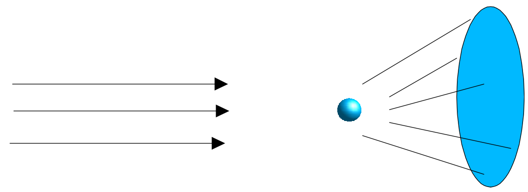
colatitude θ
azimuth φ

Detector subtends a small solid angle $d\Omega = \sin\theta \, d\theta \, d\varphi$



detects a number dS of particles, $dS = I N \sigma(\theta,\varphi) d\Omega$, where $\sigma(\theta,\varphi)$ is the **differential cross section** and
$\sigma_{tot} = \int \sigma(\theta,\varphi) d\Omega$.

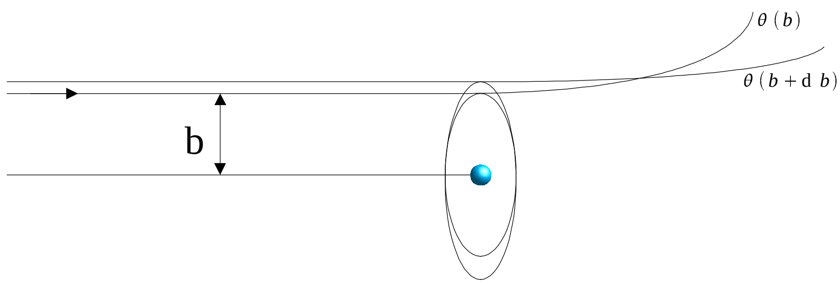
Single target particle $N=1$; central-force interaction. Let this target particle be in the center of the beam
$\Rightarrow \sigma = \sigma(\theta)$

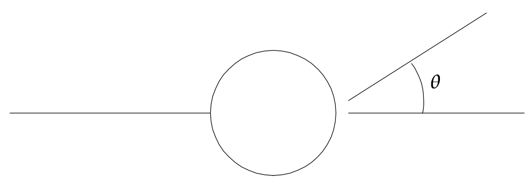


$$2\pi\,\sigma(\theta)\sin\theta\,\mathrm{d}\theta=\frac{\mathrm{d}S}{I}$$

$\mathrm{d}S$ = number of particles scattered into angles between θ and $\theta+\mathrm{d}\theta$ for any φ.

Consider an incident particle aimed some distance b (called the **impact parameter**) from the target particle.





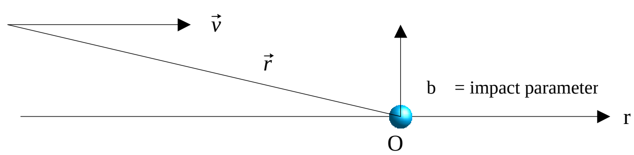
The part of the beam that lies in the ring of radius b and of width $\mathrm{d}b$ gets scattered through angles θ to $\theta+\mathrm{d}\theta$.
The ring has $2\pi b\,\mathrm{d}b$.

Number of particles scattered into $\mathrm{d}\theta$ interval =

$$\underbrace{2\pi b\,\mathrm{d}b}_{A}\cdot I$$

$$2\pi b(\theta)\,\mathrm{d}b=-2\pi\,\sigma(\theta)\sin\theta\,\mathrm{d}\theta$$

b depends on initial onditions



$$l=\mu v b$$

At $r=-\infty$:

$$E = \frac{1}{2}\,\mu\,v^2$$
$$l = b\,\sqrt{2\,\mu\,E}$$

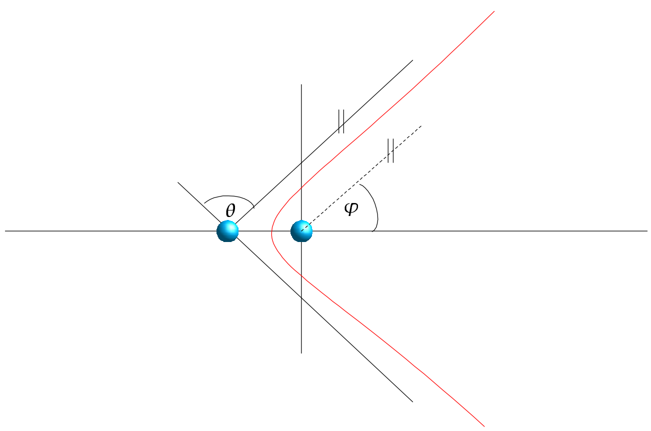
Total energy of relative motion was

$$E = \frac{1}{2}\,\mu\,\dot{r}^2 + \frac{1}{2}\,\mu\,r^2\,\dot{\varphi}^2 + V\,(r)$$
$$\frac{\mathrm{d}}{\mathrm{d}\,t} = \underbrace{\frac{l}{\mu\,r^2}}_{=\dot{\varphi}}\,\frac{\mathrm{d}}{\mathrm{d}\,\varphi}$$
$$E = \frac{l^2}{2\,\mu\,r^4}\left(\frac{\mathrm{d}\,r}{\mathrm{d}\,\varphi}\right)^2 + \frac{l^2}{2\,\mu\,r^2} + V\,(r)$$
$$\frac{\mathrm{d}\,\varphi}{\mathrm{d}\,r} = \left(\frac{\mathrm{d}\,r}{\mathrm{d}\,\varphi}\right)^{-1}$$

$\Rightarrow$ integrate motion from r_0 to r

$$\varphi\,(r) = \sqrt{\frac{l^2}{2\,\mu}}\int_{r_0}^{r}\frac{\mathrm{d}\,r}{r^2\,\sqrt{E - V\,(r) - \frac{l^2}{2\,\mu\,r^2}}}$$

Angle φ $\leftrightarrow$ scattering angle θ

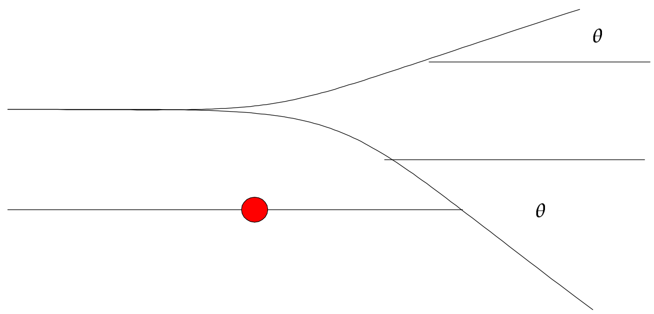


$$\theta = \pi - 2\,\varphi$$
$$\theta = \pi - 2\,b\int_{r_0}^{\infty}\frac{\mathrm{d}\,r}{r\,\sqrt{r^2\left[1 - \frac{V\,(r)}{E}\right] - b^2}}$$

where r_0 is the minimum radius = $\sqrt{\ldots}$.
Apply this recipe to the Coulomb force $V\,(r) = \pm\alpha / r$, beam particles of mass m $\Rightarrow$

Rutherford scattering cross section

$$\sigma(\theta) = \frac{\alpha^2}{4\, m^2\, v^2} \frac{1}{\sin^4 \frac{\theta}{2}}$$



$$\sigma_{tot}(\text{Coulomb}) = \int d\,\Omega\, \sigma(\theta) = \infty$$

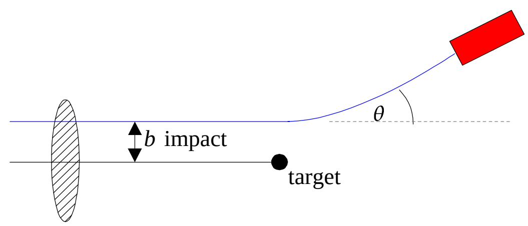
$\Rightarrow$ long-range nature of Coulomb potential

Real potential:

$$\mp \frac{\alpha}{r}\, e^{-\lambda r}$$

screened Coulomb potential

$$\sigma(\theta, \varphi) = \frac{d\,\sigma}{d\,\Omega} = \frac{\#\text{ particles scattered into } d\,\Omega \text{ per unit time}}{\#\text{ particles incoming per unit time and unit source}}$$



Differential cross section („differenzieller Wirkungsquerschnitt“), solid angle („Raumwinkel“)

5.2 Linear Oscillations

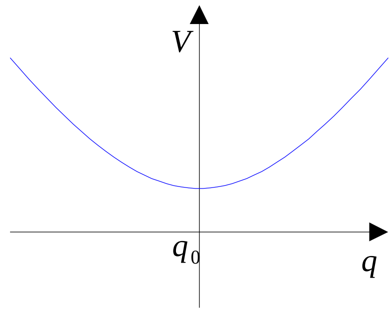
5.2.1 Linear Approximation, small vibrations

q_0 = stable equilibrium point of a Lagrangian system in one freedom

Potential energy is:

$$V(q) = V(q_0) + (q - q_0)\, V'(q_0) + \frac{1}{2}(q - q_0)^2\, V''(q_0) + \dots \text{ (Taylor)}$$

$$V'(q_0) = 0$$



$$x \equiv q - q_0 \Rightarrow$$

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

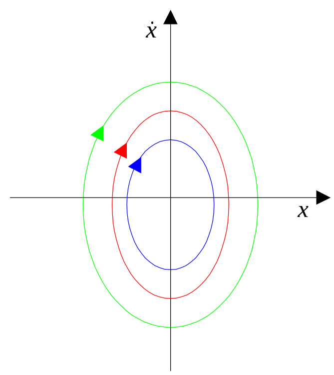
$$\Rightarrow \ddot{x} + \omega^2 x = 0 , \omega = \sqrt{\frac{k}{m}}$$

$k = V''(q_0)$ **force constant** or **effective spring constant**

$$x(t) = a \cos \omega t + b \sin \omega t = C \cos(\omega t + \delta) = \mathrm{Re}(\alpha e^{i \omega t}) = \alpha e^{i \omega t} + \alpha^* e^{-i \omega t}$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x = \frac{1}{2} \underbrace{m C^2 \omega^2}_{= k C^2} \sin^2(\omega t + \delta) + \frac{1}{2} k C^2 \cos^2(\omega t + \delta) = \frac{1}{2} k C^2 = const$$

$$\frac{\dot{x}^2}{\omega^2 C^2} \frac{x^2}{C^2} = 1$$



Origin: $x = \dot{x} = 0$

$$L = \frac{1}{2} m_{\alpha \beta}(q) \dot{q}^\alpha \dot{q}^\beta - V(q)$$

$V(q)$ has minimum at same q_0 :

$$\left. \frac{\partial V}{\partial q^\beta} \right|_{q_0} = 0 \quad \forall \beta = 1, \dots, n$$

q_0 is stable equilibrium point

$$V(x) \approx \frac{1}{2} K_{\alpha \beta} x^\alpha x^\beta$$

$$K_{\alpha\beta} = \left.\frac{\partial^2 V}{\partial x^\alpha \, \partial x^\beta}\right|_{x=0}$$
$$x^\alpha = q^\alpha - q_0^\alpha$$

$x = 0$ is minimum $\Rightarrow$ $K_{\alpha\beta}$ forms a positive Matrix, i.e.
$V(x) \equiv K_{\alpha\beta} x^\alpha x^\beta \geq 0 \quad \forall \; x \neq 0$

$\overleftrightarrow{K}$ is called **force constant matrix**
$m_{\alpha\beta}(q) = M_{\alpha\beta} + R_{\alpha\beta\gamma} x^\gamma + \ldots$
$T > 0 \Rightarrow M_{\alpha\beta}$ is positive definite,
$M_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta > 0 \quad \forall \; \dot{x} > 0$

$\Rightarrow$ to lowest order in x
$$L = \frac{1}{2} M_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta - \frac{1}{2} K_{\alpha\beta} x^\alpha x^\beta \Rightarrow$$

$$\ddot{x}^\alpha + \Lambda^\alpha_\beta x^\beta = 0$$

$$\Lambda^\alpha_\beta = M^{-1}_{\alpha\gamma} K_{\gamma\beta}$$

$$L = \frac{1}{2}(\dot{\vec{x}}, \overleftrightarrow{M} \dot{\vec{x}}) - \frac{1}{2}(\vec{x}, \overleftrightarrow{K} \vec{x})$$

Note: *M^{-1} exists since M is positive definite.*

Equations of n coupled oscillators:
$L \in T_{q_0} \mathbb{Q}$, $\vec{x} = (x^\alpha)$ are directions in $T_{q_0} \mathbb{Q}$, and $(\vec{x}, \vec{y}) = x_\alpha y_\alpha$

$$\ddot{\vec{x}} + \overleftrightarrow{\Lambda} \vec{x} = 0$$

$$\overleftrightarrow{\Lambda} = \overleftrightarrow{M}^{-1} \vec{K}$$

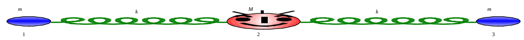
Solutions are called **normal modes** $\Leftrightarrow$ Eigenvectors and eigenvalues of $\overleftrightarrow{\Lambda}$
$\vec{\Lambda} \vec{x} = \lambda \vec{x}$
$\Rightarrow \ddot{\vec{x}} = -\lambda \vec{x} = -\omega^2 \vec{x}$

Normal frequencies ω. The eigenvectors oscillate like independent one-freedom harmonic oscillators and thus point along the normal modes. The ω are square roots of the eigenvalues of $\overleftrightarrow{\Lambda}$.
ω real $\Rightarrow \lambda \geq 0$
$\lambda = 0$: $\ddot{\vec{x}} = 0 \Rightarrow \vec{x} = \vec{A} t + \vec{B}$ force free motion of the whole system

Let $\vec{a}$ be an eigenvector:
$\overleftrightarrow{\Lambda} \vec{a} = \omega^2 \vec{a}$

$$\Rightarrow \vec{x}(t) = \vec{a}\left(\alpha\, e^{i\omega t} + \alpha^* e^{-i\omega t}\right) = 2\,\vec{a}\,\{A\cos\omega t + B\sin\omega t\}$$
$$[A = \mathrm{Re}\,\alpha,\ B = -\mathrm{Im}\,\alpha]$$
$$\vec{x}(t) = \sum_{\gamma=1}^{n} \vec{a}_\gamma\left[\alpha_\gamma\, e^{i\omega_\gamma t} + \alpha_\gamma^* e^{-i\omega_\gamma t}\right]$$
$$\alpha = \mathrm{Re}\,\alpha + i\,\mathrm{Im}\,\alpha = \alpha_R + i\,\alpha_I;$$
$$(\alpha_R + i\,\alpha_I)(\cos\omega t + i\sin\omega t) + (\alpha_R - i\,\alpha_I)(\cos\omega t - i\sin\omega t) = 2\,\alpha_R\cos\omega t - 2\,\alpha_I\sin\omega t$$

Example: Idealized linear classical water molecule



let equilibrium distance between neighboring atoms be b

$$x_2^0 - x_1^0 = b,\ x_3^0 - x_2^0 = b$$
$$V(x_1, x_2, x_3) = \frac{k}{2}\left[(x_2 - x_1 - b)^2 + (x_3 - x_2 - b)^2\right]$$
$$q_i = x_i - x_i^0$$
$$V(q_1, q_2, q_3) = \frac{k}{2}\left[(q_2 - q_1)^2 + (q_3 - q_2)^2\right]$$
$$T(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_3^2\right) + \frac{M}{2}\,\dot{x}_2^2 = \frac{m}{2}\left(\dot{q}_1^2 + \dot{q}_3^2\right) + \frac{M}{2}\,\dot{q}_2^2$$
$$L(q, \dot{q}) = \frac{1}{2}\sum_{i,j=1}^{3}\left[M_{ij}\,\dot{q}_i\,\dot{q}_j - K_{ij}\,q_i\,q_j\right]$$
$$\overset{\leftrightarrow}{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix},\ \overset{\leftrightarrow}{M}^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{M} & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix},\ \overset{\leftrightarrow}{K} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$
$$V = \frac{k}{2}\left[2\,q_2^2 - q_2\,q_1 - q_1\,q_2 + q_1^2 + q_3^2 - q_3\,q_2 - q_2\,q_3\right] = \frac{k}{2}\sum_{i,j} K_{ij}\,q_i\,q_j$$
$$\overset{\leftrightarrow}{\Lambda} = \overset{\leftrightarrow}{M}^{-1}\,\overset{\leftrightarrow}{K} = k\begin{pmatrix} \frac{1}{m} & -\frac{1}{m} & 0 \\ -\frac{1}{M} & +\frac{2}{M} & -\frac{1}{M} \\ 0 & -\frac{1}{m} & \frac{1}{m} \end{pmatrix}$$
$$\overset{\leftrightarrow}{\Lambda}\,\vec{q} = \omega^2\,\vec{q} \Leftrightarrow (\overset{\leftrightarrow}{\Lambda} - \omega^2\,\overset{\leftrightarrow}{E})\,\vec{q} = 0$$

$$\det\left(\overset{\leftrightarrow}{\Lambda} - \omega^2 \overset{\leftrightarrow}{E}\right) = 0 = \begin{vmatrix} \frac{k}{m} - \omega^2 & -\frac{k}{m} & 0 \\ -\frac{k}{M} & \frac{2\,k}{M} - \omega^2 & -\frac{k}{M} \\ 0 & -\frac{k}{m} & \frac{k}{m} - \omega^2 \end{vmatrix} = \begin{pmatrix} \lambda \equiv \frac{k}{m} \\ \mu \equiv \frac{k}{M} \end{pmatrix}$$

$$= \begin{vmatrix} \lambda - \omega^2 & -\lambda & 0 \\ -\mu & 2\,\mu - \omega^2 & -\mu \\ 0 & -\lambda & \lambda - \omega^2 \end{vmatrix} = \lambda \begin{vmatrix} -\mu & -\mu \\ 0 & \lambda - \omega^2 \end{vmatrix} + (\lambda - \omega^2) \begin{vmatrix} 2\,\mu - \omega^2 & -\mu \\ -\lambda & \lambda - \omega^2 \end{vmatrix}$$

$$= -\lambda\,\mu\,(\lambda - \omega^2) + (\lambda - \omega^2)\left[(\lambda - \omega^2)(2\,\mu - \omega^2) - \lambda\,\mu\right]$$

$$= (\lambda - \omega^2)\underbrace{\left[(\lambda - \omega^2)(2\,\mu - \omega^2) - 2\,\lambda\,\mu\right]}_{=\,\omega^2\,(\omega^2 - 2\,\mu - \lambda)}(\lambda - \omega^2)\,\omega^2\,(\omega^2 - 2\,\mu - \lambda) = 0$$

$\Rightarrow$ Normal frequencies ω_γ

$$\omega_1 = \sqrt{\frac{k}{m}}\,,\ \omega_2 = \sqrt{k\left(\frac{1}{m} + \frac{2}{M}\right)}\,,\ \omega_3 = 0$$

$$\left(\Lambda - \omega_\lambda^2\right)\vec{q} = 0$$

$$\vec{q_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \vec{q_2} = \begin{pmatrix} 1 \\ -2\,\frac{m}{M} \\ 1 \end{pmatrix},\ \vec{q_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\alpha$$

$$(q_{1\,1} = 1)$$

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\left\{\alpha_1\,e^{i\,\omega_1\,t} + \alpha_1^*\,e^{-i\,\omega_1\,t}\right\} + \begin{pmatrix} 1 \\ -2\,\frac{m}{M} \\ 1 \end{pmatrix}\left\{\alpha_2\,e^{i\,\omega_2\,t} + \alpha_2^*\,e^{-i\,\omega_2\,t}\right\} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(a + v\,t)$$

$$q_1(0) = -A\,,\ q_2(0) = A\,\frac{m}{M}\,,\ q_3(0) = 0$$

$$\dot{q}_\gamma(0) = 0 \qquad \forall\ \gamma$$

$$\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = \frac{1}{2}\,A \begin{pmatrix} \cos\,\omega_1\,t + \cos\,\omega_2\,t \\ -2\,\frac{m}{M}\cos\,\omega_2\,t \\ -\cos\,\omega_1\,t + \cos\,\omega_2\,t \end{pmatrix}$$

$$L = \frac{1}{2}\left(\dot{\vec{x}}\,,\ \overset{\leftrightarrow}{M}\,\dot{\vec{x}}\right) - \frac{1}{2}\left(\vec{x}\,,\ \overset{\leftrightarrow}{K}\,\vec{x}\right)$$

$$\ddot{\vec{x}} = +\overset{\leftrightarrow}{\Lambda}\,\vec{x} = 0$$

$$\overset{\leftrightarrow}{\Lambda} = \overset{\leftrightarrow}{M}^{-1}\,\overset{\leftrightarrow}{K}$$

$$\overset{\leftrightarrow}{\Lambda}\,\vec{x} = \lambda\,\vec{x} \quad ; \quad \lambda \equiv \omega^2$$

1) Given an arbitrary square matrix $A_{n\times n}$, $a_{ij}\in\mathbb{R}$. Then A possesses exactly n eigenvalues $\lambda_i\in\mathbb{C}$. If $\lambda_i\neq\lambda_j$, $\forall\ i,\ j$, then there exist exactly n linearly independent eigenvectors.
2) Given a real symmetric matrix. All eigenvalues are real. If $\lambda_i\neq\lambda_j$, $\forall\ i,j$, the eigenvectors are orthogonal to each other. If an eigenvalue is p-fold degenerated, there exists p linearly independent eigenvectors that may be orthogonalized (e.g. by the Gram-Schmid-Procedure). $\Rightarrow$ **Orthogonal eigenvectors**
3) Given A, B. A, B are real symmetric matrices ($n\times n$). If $A\,B\neq B\,A \Rightarrow A\,B$ is not symmetric. Therefore, its eigenvectors are not orthogonal even though $\lambda_i\neq\lambda_j\quad\forall\ i,\ j$

$$\overset{\leftrightarrow}{\Lambda}\,\vec{x}=\omega^2\,\vec{x}$$

$$\Lambda=\underbrace{\begin{pmatrix}\frac{1}{m}&0&0\\0&\frac{1}{M}&0\\0&0&\frac{1}{m}\end{pmatrix}}_{M^{-1}}\underbrace{\begin{pmatrix}\frac{k}{m}&-\frac{k}{m}&0\\-\frac{k}{M}&\frac{2\,k}{M}&-\frac{k}{M}\\0&-\frac{k}{m}&\frac{k}{m}\end{pmatrix}}_{K}$$

$$\omega_1=\sqrt{\frac{k}{m}},\quad\omega_2=\sqrt{k\left(\frac{1}{m}+\frac{2}{M}\right)},\quad\omega_3=0$$

$$\vec{q}_1=c_1\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \vec{q}_2=c_2\begin{pmatrix}1\\-2\,\frac{m}{M}\\1\end{pmatrix},\quad q_3=c_3\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\overset{\leftrightarrow}{M}^{-1}\,\overset{\leftrightarrow}{K}\,\vec{x}=\omega^2\,\vec{x}$$

$$\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\overset{\leftrightarrow}{K}\,\vec{x}=\omega^2\,\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\vec{x}$$

$$\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\overset{\leftrightarrow}{K}\,\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\overset{\leftrightarrow}{M}^{\frac{1}{2}}\,\vec{x}=\omega^2\,\overset{\leftrightarrow}{M}^{\frac{1}{2}}\,\vec{x}$$

$$\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\overset{\leftrightarrow}{K}\,\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\vec{y}=\omega^2\,\vec{y}$$

$$\overset{\leftrightarrow}{M}^{-\frac{1}{2}}\,\overset{\leftrightarrow}{K}\,\overset{\leftrightarrow}{M}^{-\frac{1}{2}}=\begin{pmatrix}\frac{k}{m}&-\frac{k}{\sqrt{m\,M}}&0\\-\frac{k}{\sqrt{m\,M}}&\frac{2\,k}{M}&-\frac{k}{\sqrt{m\,M}}\\0&-\frac{k}{\sqrt{m\,M}}&\frac{k}{m}\end{pmatrix}$$

$$\vec{y}_1=c_1\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \vec{y}_2=c_2\begin{pmatrix}1\\-2\,\sqrt{\frac{m}{M}}\\1\end{pmatrix},\ \vec{y}_3=c_3\begin{pmatrix}1\\\sqrt{\frac{M}{m}}\\1\end{pmatrix}$$

5.2.2 Forced and Damped Oscillations

One-freedom case: undamped harmonic oscillator driven by an additional external time-dependent force $F(t)$.
If the force depends only on time, it can easily be fit into the Langrangian formalism. The time-dependent Lagrangian

$L'(q, \dot{q}, t) = L(q, \dot{q}) + q F(t)$ (L' does not mean „derivative“)

$$m \ddot{q} = -\frac{\partial V}{\partial q} + F(t)$$

Oscillator:

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 + q F(t)$$

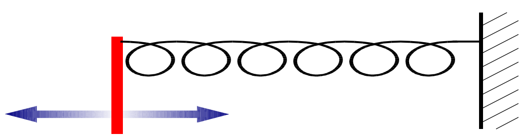
$$\ddot{q} + \omega_0^2 q = f(t); (*)$$

$$f(t) = \frac{F(t)}{m}, \omega_0 = \sqrt{\frac{k}{m}}$$

forced or **driven oscillator**

Assume: Driving force $F(t) = F_0 \cos \Omega t$

ω : angular velocity of the oscillator
Ω : angular velocity of the driving force



A particular solution has the form
$C \cos(\Omega t + \gamma)$

$$(*) \Rightarrow \gamma = 0, C = \frac{f_0}{\omega_0^2 - \Omega^2}$$

General solution of (*) for $\Omega \neq \omega_0$:

$$q(t) = A \sin(\omega_0 t + \delta) + \frac{f_0 \cos \Omega t}{\omega_0^2 - \Omega^2}$$

A, δ depend on the initial conditions.
Ω -term is in phase with driving force if $\Omega < \omega_0$. If $\Omega > \omega_0$, $q(t)$ oscillates out of phase (phase factor π) with driving force.

$\Omega = \omega_0$:

$$q(t) = A \sin(\omega_0 t + \delta) + \frac{f_0 t \sin \omega_0 t}{2 \omega_0}$$

Forced Damped Oscillator

$$\ddot{q} + 2 \beta \dot{q} + \omega_0^2 q = f(t)$$

$\beta = \frac{b}{m}$ is the damping factor

$(F = \frac{1}{2} b \dot{q}^2 , \vec{F}_D = -b \dot{q})$

$f(t) = f_0 \cos \Omega t$
$q_h(t) = \alpha e^{(-\beta + i \omega) t} \qquad \alpha \in \mathbb{C}$
$\omega = \sqrt{\omega_0^2 - \beta^2}$ is the so-called **natural frequency**

Real solution:
$e^{-\beta t} \left[\operatorname{Re} \alpha \cos \omega t - \operatorname{Im} \alpha \sin \omega t \right]$
$\alpha = a e^{i \delta}$ where $\alpha , \delta \in \mathbb{R}$
$\Rightarrow \operatorname{Re} q_n(t) = a e^{-\beta t} \cos(\omega t + \delta)$
$\Rightarrow$ Complex phase δ of α is the **physical phase** of the oscillation.

The complex solution:
$f = f_0 e^{i \Omega t}$

$$q(t) = \alpha e^{-(\beta + i \omega) t} + \Gamma e^{i \Omega t}$$

(first term: **transient response**;
second term: **steady state response**;)

$$\Gamma = \frac{f_0}{\omega_0^2 - \Omega^2 + 2 i \beta \Omega}$$

Phase difference γ between the driving force to $e^{i \Omega t}$ and Γ is

$$\gamma = \arctan \frac{\operatorname{Im} \left(\frac{f_0}{\Gamma} \right)}{\operatorname{Re} \left(\frac{f_0}{\Gamma} \right)}$$

$$\frac{f_0}{\Gamma} = \omega_0^2 - \Omega^2 + 2 i \beta \Omega \Rightarrow \tan \gamma = \frac{2 \beta \Omega}{\omega_0^2 - \Omega^2}$$

Steady state amplitude:

$$| \Gamma | = \frac{|f_0|}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4 \beta^2 \Omega^2}}$$

$| \Gamma |$ has maximum when
$\Omega = \sqrt{\omega_0^2 - 2 \beta^2}$

maximum value is

$$\frac{|f_0|}{2 \beta}$$

6. Rigid Bodies

6.1 Introduction

6.1.1 Rigidity and Kinematics

Definition: A **rigid body** is an extended collection of point particles so that the distance between any two of them remains constant.

The configuration of the body:

- by position of an arbitrary point (3 coordinate)
- by its orientation relative to the axes of an inertial system (3 angles)

$\Rightarrow \dim \mathbb{Q} = 6$

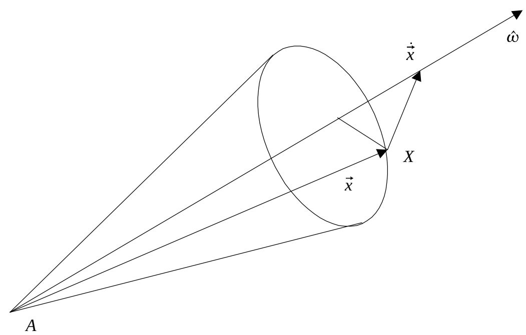
Common: fixed point = pivot

If sum of the external forces vanishes $\Rightarrow$ center of mass does not accelerate, may serve as pivot even if body has no fixed point
Henceforth: assume that there is an **inertial point** A in the body

Describe motion of a rigid body in the inertial frame. Rigidity implies:

(1)that there is an instantaneous axis of rotation $\hat{\omega}$, i.e. a line in the body that passes through the origin A and is instantaneously at rest.

(2)all points in the body move at right angles to $\hat{\omega}$ at speeds proportional to their distance from this instantaneous axis



$$\text{Inertial point } A = \begin{cases} \text{center of mass} \\ \text{pivot} \end{cases}$$

$X \neq A$, $\vec{x} \cdot |\vec{x}| = const$

$$\frac{\mathrm{d}}{\mathrm{d}\, t}(x^2) = 2\, \vec{x}\, \dot{\vec{x}} = 0$$

$\Rightarrow$ There mus be a line $\hat{\omega}$ in the body that is instantaneously stationary.
$\hat{\omega}$ ist instantaneous axis of rotation , speed of X is $\dot{x} = \omega\, r_x$.
ω : **instantaneous angular speed** or **rate of rotation**

Definition: **Instananeous angular velocity vector**:

$$\vec{\omega} : \frac{\vec{\omega}}{|\vec{\omega}|} = \hat{\omega} \; ; \; |\vec{\omega}| = \omega \; ;$$

$$\Rightarrow \dot{\vec{x}} = \vec{\omega} \times \vec{x}$$

6.12 Kinetic Energy and Angular Momentum

In inertial system, origin A. Mass density $\mu(\vec{x})$.

$$T = \frac{1}{2} \int \dot{x}^2 (\vec{x}) \, \mu(\vec{x}) \, \mathrm{d}^3 x = \frac{1}{2} \int \dot{x}^2 \, \mathrm{d} m \; ;$$

$\dot{x}_k = \varepsilon_{k\,l\,m} \, \omega_l \, x_m$; (implied summation)

$$\dot{x}^2 = \dot{x}_k \, \dot{x}_k = \varepsilon_{k\,l\,m} \, \omega_l \, x_m \, \varepsilon_{k\,i\,j} \, \omega_i \, x_j = (\delta_{l\,i} \, \delta_{m\,j} - \delta_{l\,j} \, \delta_{m\,i}) \, \omega_l \, \omega_i \, x_m \, x_j = \omega_l \, (\delta_{l\,i} \, x^2 - x_l \, x_i) \, \omega_i$$

$$T = \frac{1}{2} \omega_l \left[\int \left(\delta_{l\,i} \, x^2 - x_l \, x_i \right) \mathrm{d} m \right] \omega_i = \frac{1}{2} \omega_l \, I_{l\,i} \, \omega_i = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \, \vec{\omega} \; ;$$

$$I_{l\,i} \equiv \int (\delta_{l\,i} \, x^2 - x_l \, x_i) \, \mathrm{d} m$$

are the elements of the **inertia tensor** or **inertia matrix** (G: Trägheitstensor)

$$T = \frac{1}{2} \vec{v} \cdot m \, \vec{v}$$

$I > 0 \Rightarrow \exists \, I^{-1}$; $\overleftrightarrow{I}$ is symmetric

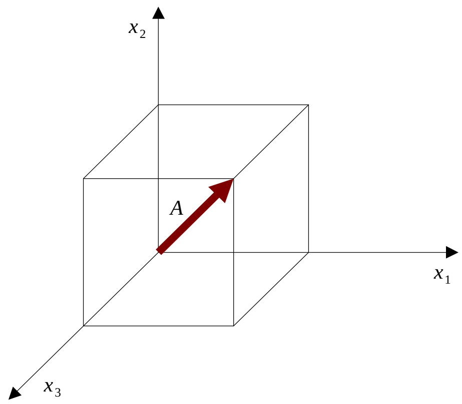
$\Rightarrow$ diagonalize by orthogonal transformation $\Rightarrow$ there is an orthogonal coordinate system whose basis vectors are eigenvectors of $\overleftrightarrow{I}$. In this coordinate system $\overleftrightarrow{I}$ is diagonal: **principal axis system** (G: Hauptachsensystem), the eigenvalues of $\overleftrightarrow{I}$ are called **principal moments** or **moments of inertia** (G: Hauptträgheitsmomente).

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$$T = \frac{1}{2} (I_1 \, \omega_1^2 + I_2 \, \omega_2^2 + I_3 \, \omega_3^2)$$

ω_k are the components of $\vec{\omega}$ in the principal axis system.

Example: Uniform cube of side s ; pivot A is at a corner.



$$I_{11} = I_{22} = I_{33} = \mu \int_0^s \mathrm{d}\, x_1 \int_0^s \mathrm{d}\, x_2 \int_0^s \mathrm{d}\, x_3 \, (x_2^2 + x_3^3) = \frac{2}{3}\, \mu\, s^5$$

$$I_{12} = I_{23} = I_{31} = -\mu \int_0^s \mathrm{d}\, x_1 \int_0^s \mathrm{d}\, x_2 \int_0^s \mathrm{d}\, x_3\, x_1\, x_2 = -\frac{1}{4}\, \mu\, s^5$$

$$I = \frac{2}{3}\, M\, s^2 \begin{pmatrix} 1 & -\frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & 1 & -\frac{3}{8} \\ -\frac{3}{8} & -\frac{3}{8} & 1 \end{pmatrix};$$

$M \equiv \mu\, s^3 =$ mass of the cube

Eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}, \; \alpha = -\frac{3}{8}$$

$$\det\,(A - \lambda\, E) = 0 = \det \begin{pmatrix} 1-\lambda & \alpha & \alpha \\ \alpha & 1-\alpha & \alpha \\ \alpha & \alpha & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^3 - 3\,\alpha^2\,(1-\lambda) + 2\,\alpha^3 = (1-\lambda-\alpha)^2\,(1-\lambda+2\,\alpha)$$

$$\lambda_1 = 2\,\alpha + 1 = \frac{1}{4} \Rightarrow \vec{q}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = 1 - \alpha = \frac{11}{8} \Rightarrow \vec{q}_2 \perp \vec{q}_3 \perp \vec{q}_1$$

$$\Rightarrow I_1 = \frac{1}{6}\, M\, s^2\,,\; I_2 = I_3 = \frac{11}{12}\, M\, s^2\,;$$

Angular Momentum

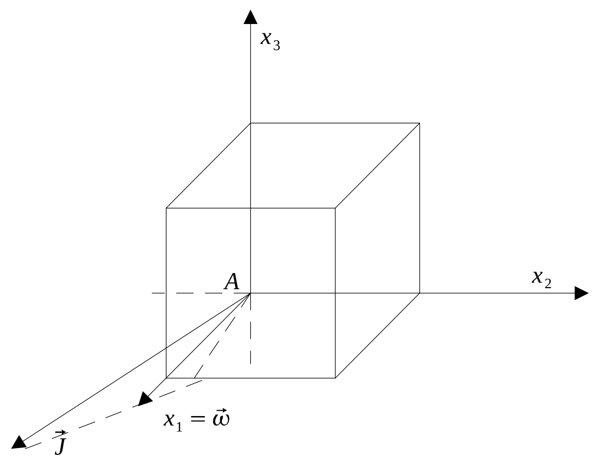
The angular momentum of the rigid body is

$$\vec{J} = \int \mu\,(\vec{x})\, \vec{x} \times \dot{\vec{x}}\, \mathrm{d}^3\, x = \int \vec{x} \times \dot{\vec{x}}\, \mathrm{d}\, m$$

$$\left[\vec{x}\times(\vec{\omega}\times\vec{x})\right]_i=\varepsilon_{i\,j\,k}\;x_j\;\varepsilon_{k\,l\,m}\;\omega_l\;x_m=(\delta_{i\,l}\;\delta_{j\,m}-\delta_{i\,m}\;\delta_{j\,l})\;x_j\;\omega_l\;x_m=(\delta_{i\,l}\;x^2-x_i\;x_l)\;\omega_l$$

$J_i=I_{i\,l}\;\omega_l$ or $\vec{J}=\overset{\leftrightarrow}{I}\;\vec{\omega}$

Example:



$$\vec{\omega}=\begin{pmatrix}\Omega\\0\\0\end{pmatrix}$$

$$\vec{J}=\frac{2}{3}\;M\;\varepsilon^2\begin{bmatrix}1&\alpha&\alpha\\\alpha&1&\alpha\\\alpha&\alpha&1\end{bmatrix}\begin{bmatrix}\Omega\\0\\0\end{bmatrix}=M\;\varepsilon^2\;\Omega\begin{pmatrix}\frac{2}{3}\\-\frac{1}{4}\\-\frac{1}{4}\end{pmatrix}$$

6.1.3 Dynamics

The equation for rigid body motion was (2.4.5):

$N_Z=\dot{\vec{J}}_Z$

Z : both vectors must be calculated about the center of mass or about a pivot

Space and Body System

1) A noninertial system, called the **body system** B , is fixed in the body and moves with it.

> ***Note**: We shall denote column vectors that contain the components of a vector in a given coordinate system without an arrow. Since the components x_B of the position vector $\vec{x}$ of any point in the body are fixed, $\dot{x}_B=0$.*

2) An inertial system, called the **space system** S . We write x_S for the representation of $\vec{x}$ in S and $\dot{x}_S\neq 0$.

Both origins are at A (inertial point). For some time both coordinates coincide.
Suppose some such coordinate system is specified. Then $\vec{x}$ is fixed in the body, $\dot{\vec{x}}$ is the rate of

change viewed from the specified system and $\vec{\omega}$ is the angular velocity of the body with respect to the specified system.
In particular, if the system S that coincides instantaneously with B, $\dot{\vec{x}} = \vec{\omega} \times \vec{x}$ reads
$\dot{x_S} = \omega_B \times x_B$ (*)

We can extend (*) to an arbitrary vector $\vec{s}$ in the body, one that may be moving in the body system B. Its velocity $\dot{s_S}$ relative to the inertial system S is the velocity $\omega \times s_B$ it would have if it were fixed in B and velocity $\dot{s_B}$ relative to B:

$$\dot{s_S} = \omega_B \times s_B + \dot{s_B} \quad (**)$$

Tells us how to transform velocity vectors between B and S.

If $\dot{s}$ in this equation is angular velocity,
$$\dot{\omega}_S = \underbrace{\omega_B \times \omega_B}_{=0} + \dot{\omega}_B = \dot{\omega}_B$$

$\Rightarrow$ The body and space representations of $\vec{\omega}$ and $\dot{\vec{\omega}}$ are identical.

Dynamical Equations

If (**) is applied to $\dot{s} \rightarrow \dot{J}$
$\dot{J}_S = \omega \times J_B + \dot{J}_B = \omega \times (I_B\ \omega) + I_B\ \dot{\omega}$
$\Rightarrow$ Equations of motion reads $N_S = \omega \times (I_\omega) + I\ \dot{\omega}$.

In the principal axis system, this reads
$N_1 = (I_3 - I_2)\ \omega_3\ \omega_2 + I_1\ \dot{\omega}_1$
$N_2 = (I_1 - I_3)\ \omega_1\ \omega_3 + I_2\ \dot{\omega}_2$
$N_3 = (I_2 - I_1)\ \omega_2\ \omega_1 + I_3\ \dot{\omega}_3$

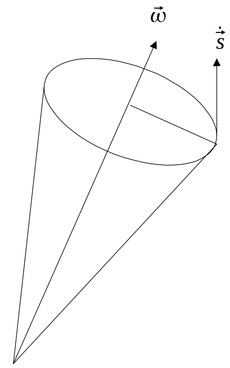
Euler Equations

Rigid bodies: 6 degrees of freedom

Inertial point, pivot:

- remains fixed
- center of mass

$$\left(\frac{\mathrm{d}\ \vec{s}}{\mathrm{d}\ t}\right)_S = \left(\frac{\mathrm{d}\ \vec{s}}{\mathrm{d}\ t}\right)_B + \vec{\omega} \times \vec{s}$$
$\vec{\omega}$ instaneous angular velocity



$$\vec{N}_Z = \frac{\mathrm{d}}{\mathrm{d}\ t_S}\left(\vec{J}_Z\right) = \frac{\mathrm{d}}{\mathrm{d}\ t_S}\left(\overleftrightarrow{I}\ \vec{\omega}\right)$$

$\vec{I}_Z$ = angular momentum

$\overleftrightarrow{I}$ = inertial tensor

Write down in body system: $\overleftrightarrow{I} = \mathrm{cent}$

$$\vec{N}_B = \frac{\mathrm{d}}{\mathrm{d}\ t_B}\ \vec{J}_B + \vec{\omega}_B \times \vec{J}_B = \overleftrightarrow{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\left(\overleftrightarrow{I}\ \vec{\omega}\right)$$

Principal axis system:

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

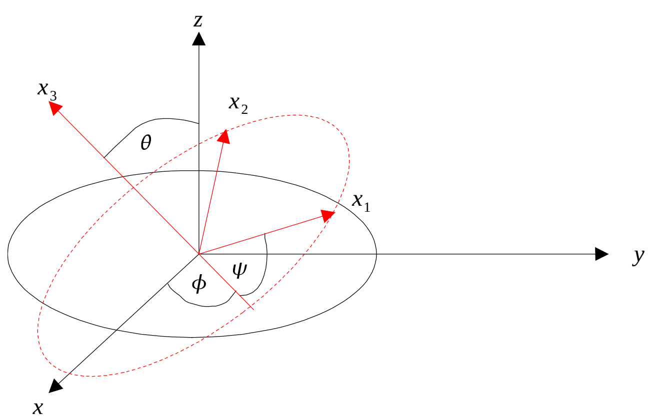
$$N_1 = \left(I_3 - I_2\right)\ \omega_3\ \omega_2 + I_1\ \dot{\omega}_1$$
$$N_2 = \left(I_1 - I_3\right)\ \omega_1\ \omega_3 + I_1\ \dot{\omega}_2$$
$$N_3 = \left(I_2 - I_1\right)\ \omega_2\ \omega_1 + I_1\ \dot{\omega}_3$$

Euler Angles

let x , y , z denote axes of *space* system

x_1 , x_2 , x_3 ~ *body* system

ϕ , ψ , θ



$$\left(x\ ,\ y\ ,\ z\right) \leftrightarrow \left(x_1\ ,\ x_2\ ,\ x_3\right)$$

Rotations through finite angles: not commutative!
Angular velocity vectors are commutative.

Therefore, $\vec{\omega} = \dot{\overleftrightarrow{\Theta}} + \dot{\overleftrightarrow{\Phi}} + \dot{\overleftrightarrow{\Psi}}$

Euler angular velocity vectors are directed along their instantenous axes of rotation:

$\dot{\overleftrightarrow{\Psi}} \parallel x_3$, $\dot{\overleftrightarrow{\Phi}} \parallel z$, $\dot{\overleftrightarrow{\Phi}} \parallel$ modal line of 2 planes

$\vec{\omega} = (\omega_x , \omega_y , \omega_z)$ in space system

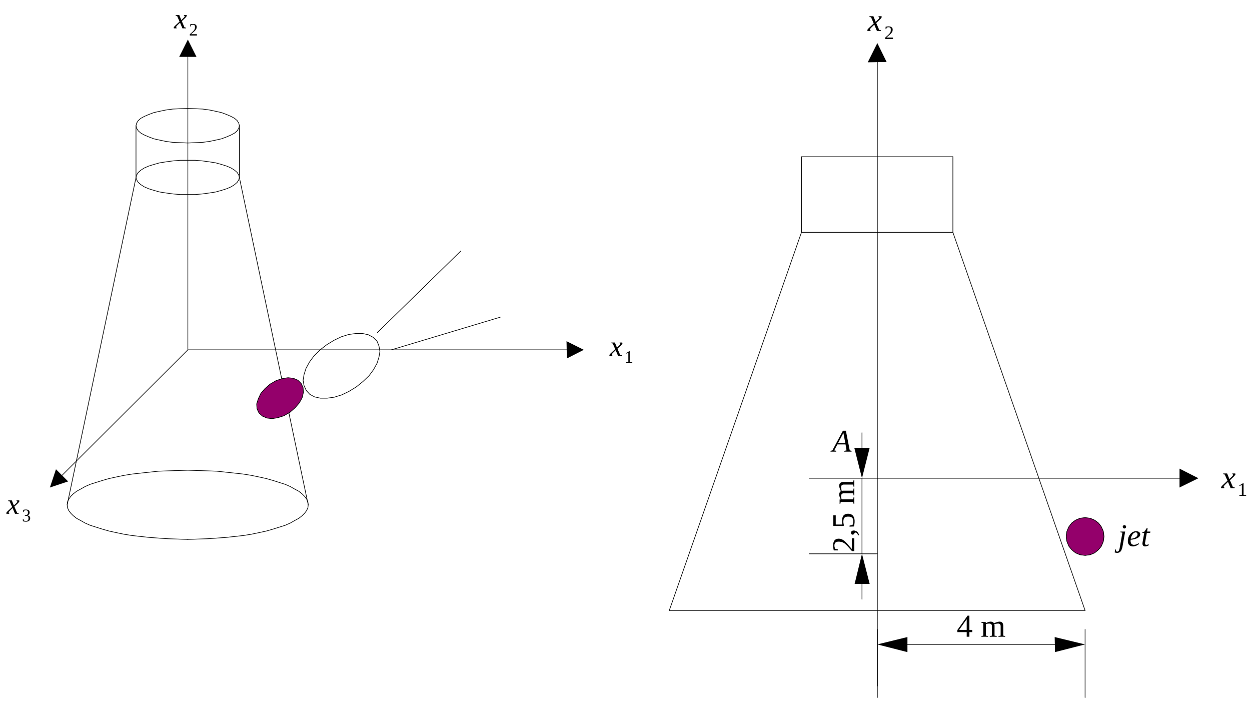
$\vec{\omega} = (\omega_1 , \omega_2 , \omega_3)$ in body system

$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$

$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$

$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$

6.1.4 Example: Jet-Driven Space Craft



$m = 2000$ kg

radii of gyration $\varrho_{x_1} = \varrho_{x_3} = 1{,}5$ m ; $\varrho_{x_2} = 1.75$ m

$\Leftrightarrow$ the principal moment of inertia in x-direction is $I_1 = m\, \varrho_{x_1}^2$, $I_2 = m\, \varrho_{x_2}^2$, $I_3 = m\, \varrho_{x_3}^2$

Initially, the spacecraft is at rest. Then, the jet fires for 1 s. It has a thrust of 25 N that acts parallel to the x_3 -axis.

- Axis of precession?
- Angular velocity and the rates of precession after the jet has stopped?

1st part: let $t = 0$ be the time when the rocket is turned on. We use the *body system*.

$$\vec{M} = \overleftrightarrow{I}\, \dot{\vec{\omega}} + \vec{\omega} \times (\overleftrightarrow{I}\, \vec{\omega}) = I_1\, \dot{\omega}_1\, \vec{e}_1 + I_2\, \dot{\omega}_2\, \vec{e}_2 + I_3\, \dot{\omega}_3\, \vec{e}_3 + O(\omega^2)$$

$\omega^2 \ll \dot{\omega}$ during that 1 s.

$\vec{M} = \vec{r} \times \vec{F} = (s_1\ \vec{e}_1 - s_2\ \vec{e}_2) \times F\ \vec{e}_3$; $(s_1, s_2) = (4\ \text{m}, 2.5\ \text{m})$;

$\vec{M} = -s_2\ F\ \vec{e}_1 - s_1\ F\ \vec{e}_2$

$\Rightarrow \dot{\omega}_1 = -s_2 \frac{F}{I_1} = -C_1$

$\Rightarrow \dot{\omega}_2 = -s_1 \frac{F}{I_2} = -C_2$

$\Rightarrow \dot{\omega}_3 = 0$

$t = 0 : \omega_i(0) = 0$

$\omega_1(t) = -C_1\ t$

$\omega_2(t) = -C_2\ t$

$\omega_3(t) = 0$

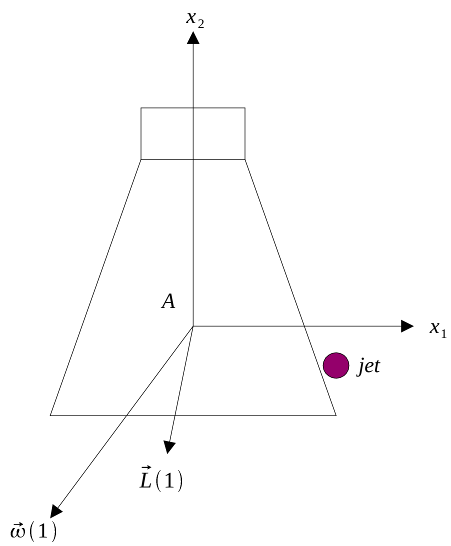
After 1 s:

$\omega_1(1) = \frac{-2.5\ \text{m} \cdot 25\ \text{N}}{2000\ \text{kg}\ (1.5\ \text{m})^2} \cdot 1\ \text{s} = -0.0139\ \text{rad/s}$

$\omega_2(1) = -0.0163\ \text{rad/s}$

$\omega^2 = \omega_1^2 + \omega_2^2 \Rightarrow |\vec{\omega}(1)| = 0.02\ \text{rad/s}$

$\tan\varphi = \frac{\omega_1}{\omega_2} = 0.85 \Rightarrow \varphi(1) = 44{,}5\ °$



The capsule is rotating around a momentary axis that points along the negative 1- and 2 direction.

$\vec{L} = I_1\ \omega_1\ \vec{e}_1 + I_2\ \omega_2\ \vec{e}_2 + I_3\ \omega_3\ \vec{e}_3$

$\vec{L}(1) = -2000\ (1.5)^2\ (0.0139)\ \vec{e}_1 - 2000\ (1.75)^2\ (0.0163)\ \vec{e}_2$

$\tan\Theta = \frac{L_1}{L_2} = 0.63 \Rightarrow \Theta(1) = 32\ °$

2. part

Motion after rocket has stopped. $t = 0$

$\vec{\omega}(1), \vec{L}(1)$ became initial values for the subsequent motion.
Remain in body system: $\vec{M} = 0$

i.e. $\left(\frac{\mathrm{d}\,\vec{L}}{\mathrm{d}\,t}\right)_{\text{Spacesystem}} = 0 \Rightarrow \vec{L}$ does not change in the *space system*.

$$I_1\,\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\,\omega_3 = 0$$
$$I_2\,\dot{\omega}_2 + (I_1 - I_3)\,\omega_1\,\omega_3 = 0$$
$$I_3\,\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\,\omega_2 = 0$$
$$I_1 = I_3 \Rightarrow \dot{\omega}_2 = 0 \Rightarrow \omega_2(t) = \omega_2(0) = -0.0163\ \text{rad/s}$$

$$A = \frac{I_2 - I_1}{I_1}\ ;\ B = \omega_2(0)\ ;\quad \Rightarrow$$

$$\dot{\omega}_1 - A\,B\,\omega_3 = 0$$
$$\dot{\omega}_3 + A\,B\,\omega_1 = 0$$

$$\omega_1 = \alpha\,e^{k\,t},\ \omega_3 = \beta\,e^{k\,t}$$
$$\Rightarrow$$
$$\omega_1(t) = C\,\cos(\gamma\,t + \theta)$$
$$\omega_3(t) = -C\,\sin(\gamma\,t + \theta),\ \ \gamma = A\,B$$

$$t = 0:\ \ \omega_1(0) = -0.0139\ \text{rad/s},\ \ \omega_3(0) = 0$$

$$\Rightarrow\ \ \theta = 0$$

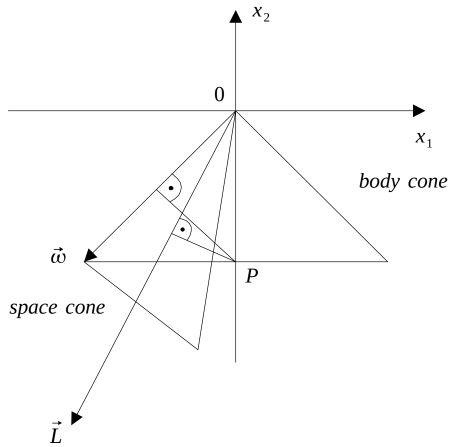
$$\omega_1(t) = \omega_1(0)\cos\gamma\,t$$
$$\omega_3(t) = \omega_1(0)\sin\gamma\,t$$
$$\gamma = A\,B$$

$$\omega_1^2(t) + \omega_3^2(t) = \omega_1^2(0)$$

Rotation about x_2-axis in the (x_1, x_2)-plane, $\vec{\omega}$ also maintains its initial angular displacement φ with respect to the x_2-axis. The cone that traces out is called the **body cone** („Polkegel“): motion is with respect to (x_1, x_2, x_3) fixed in the body.

Frequency of rotation around the x_2-axis: $P = A\,B = -0.0059\ \text{rad/s}$; $\vec{L}$ stays fixed.

Inertial system: $\vec{\omega}$ rotates around $\vec{L}$ with a frequency of rotation Ω . This motion also generates a cone called the **space cone** („Spurkegel“).



These cones have always a common line of contact. Thus, $\vec{\omega}$ acts as an instantaneous axis and the general motion can be characterized by noting that the body cone rolls on the fixed space cone.

Calculate rotation frequency Ω :
P will turn through a distance $\mathrm{d}\, s = a\ \omega\ \mathrm{d}\, t$ about $\vec{\omega}$. It will also turn through a distance $\mathrm{d}\, s = b\ \Omega\ \mathrm{d}\, t$ with respect to L . $\Rightarrow$

$$a\ \omega = b\ \Omega \Rightarrow \Omega = \omega\ \frac{\sin\ \varphi}{\sin\ \theta} = 0.2616\ \frac{\mathrm{rad}}{\mathrm{s}}\ .$$

6.2 Lagrange Equations for Rigid Bodies

$$\frac{\mathrm{d}}{\mathrm{d}\ t}\,(\overset{\leftrightarrow}{I}\ \vec{\omega}) = \vec{M}$$

efficient only if $\vec{M} = 0$ or fixed in body.

We consider pivoted bodies and take the origin of the body system in that pivot.
Pivot $= X$ (center of mass) $\Rightarrow$ rotational and translational motion are decoupled
Pivot = fixed $\Rightarrow$ only rotation

We choose 3 Euler angles $\phi\ , \psi\ , \theta$ as generalized coordinates. We assume that the potential energy depends only on those coordinates and on time. Then

$$L\ (\phi\ ,\ \psi\ ,\ \theta\ ,\ \dot{\phi}\ ,\ \dot{\psi}\ ,\ \dot{\theta}) = T_{rot} - U\ = \frac{\vec{\omega}\cdot(\overset{\leftrightarrow}{I}\ \vec{\omega})}{2} - U\ (\phi\ ,\ \psi\ ,\ \theta\ ,\ t)$$

In principal axes, we have, in terms of the body-system components I_i and ω_i

$$T_{rot} = \frac{I_1\ \omega_1^2 + I_2\ \omega_2^2 + I_3\ \omega_3^2}{2}$$

$$L\ (\phi\ ,\ \psi\ ,\ \theta\ ,\ \dot{\phi}\ ,\ \dot{\psi}\ ,\ \dot{\theta}) = \frac{I_1}{2}\,(\dot{\phi}\ \sin\ \theta\ \sin\ \psi + \dot{\theta}\ \cos\ \psi)^2$$

$$+\frac{I_2}{2}\,(\dot{\phi}\ \sin\ \theta\ \cos\ \psi - \dot{\theta}\ \sin\ \psi)^2 + \frac{I_3}{2}\,(\dot{\phi}\ \cos\ \theta + \dot{\psi})^2 - U\ (\phi\ ,\ \psi\ ,\ \theta\ ,\ t)$$

7. Hamiltonian Formulation of Mechanics

In some cases, a system becomes easier to understand and to discuss by choosing the generalized coordinates q^α and the generalized momenta p^α rather than q^α and $\dot{q}^\alpha$ as independent variables. Such a change in variables produces pairs of first-order rather than second-order ODEs.

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \quad (*)$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\, p_\alpha = \frac{\partial L}{\partial q^\alpha} \quad (**)$$

$$\frac{\mathrm{d}\,q^\alpha}{\mathrm{d}\,t} = \dot{q}^\alpha$$

$L, \dot{q}$ are all functions of $(q, \dot{q}, t)$ rather than of (q, p, t). Assume we can invert (*) to get $\dot{q}$ in terms of (q, p, t). The right hand side of (**) can be transformed: Replace $\dot{q}$ in $\partial L / \partial q$ by $\dot{q}^\alpha(q, p, t)$. The partial derivative of $L(q, \dot{q}, t)$ with respect to q^α is not the same as that of $\hat{L}(q, p, t) = L(q, \dot{q}(q, p, t), t)$.

$$\frac{\partial \hat{L}}{\partial q^\alpha} = \frac{\partial L}{\partial q^\alpha} + \frac{\partial L}{\partial \dot{q}^\beta}\frac{\partial \dot{q}^\beta}{\partial q^\alpha} = \frac{\partial L}{\partial q^\alpha} + p_\beta \frac{\partial \dot{q}^\beta(q, p, t)}{\partial q^\alpha}$$

$$\frac{\partial L}{\partial q^\alpha} = \frac{\partial}{\partial q^\alpha}\left[\hat{L}(q, p, t) - p_\beta\, \dot{q}^\beta(q, p, t)\right]$$

The derivative of $\hat{L}$ with respect to p_α:

$$\frac{\partial \hat{L}}{\partial p_\alpha} = \frac{\partial L}{\partial \dot{q}^\beta}\frac{\partial \dot{q}^\beta}{\partial p^\alpha} = p_\beta \frac{\partial \dot{q}^\beta}{\partial p_\alpha}$$

$$\frac{\partial}{\partial p_\alpha}\left[\hat{L}(q, p, t) - p_\beta\, \dot{q}^\beta(q, p, t)\right] = -\dot{q}^\alpha$$

Definition: $H(q, p, t) \equiv p_\beta\, \dot{q}^\beta(q, p, t) - \hat{L}(q, p, t)$ is called the **Hamiltonian function** or the **Hamiltonian**.

For many dynamical systems: $E = H = T + V$ where $L = T - V$.

$$\dot{q}^\beta = \frac{\partial H(q, p, t)}{\partial p_\beta}, \quad \dot{p}_\beta = -\frac{\partial H}{\partial q^\beta}$$

Hamilton's canonical equations: Are equations of motion in the Hamiltonian formalism.
$(q(t), p(t))$-manifold.
The term „canonical“ means *standard* or *conventional*.

Examples:

1) The Lagrangian of a particle in a potential

$$L = \frac{m}{2}(\dot{x}_1^{\,2} + \dot{x}_2^{\,2} + \dot{x}_3^{\,2}) - V(x_1, x_2, x_3, t)$$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\ \dot{x}_i$$

$$H\ (x\ ,\ p\ ,\ t) = \sum_i p_i\ \frac{p_i}{m} - \frac{m}{2}\ \sum_i \frac{p_i^2}{m^2} + V\ (x\ ,\ t) = \frac{1}{2\ m}\ (\ p_1^2 +\ p_2^2 +\ p_3^2) + V\ (x\ ,\ t) = T + V$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i$$

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}$$

$$\Leftrightarrow m\ \ddot{\vec{x}} = -\text{grad}\ V$$

2) The Lagrangian in an electromagnetic field was

$$L\ (q\ ,\ \dot{q}\ ,\ t) = \frac{1}{2}\ m\ \dot{q}^\beta\ \dot{q}^\beta - e\ \varphi\ (q\ ,\ t) + e\ \dot{q}^\beta\ A_\beta\ (q\ ,\ t)$$

($\beta = 1\ ,\ 2\ ,\ 3$ implied summations)

The generalized momentum p_γ that is **canonically conjugate** to q^γ.

$$p_\gamma = m\ \dot{q}^\gamma + e\ A_\gamma$$

This canonically conjugate momentum is not equal to the dynamical momentum $m\ \dot{q}^\gamma$ but has an additional contribution from A.

$$\Rightarrow\ \ H\ (q\ ,\ p\ ,\ t) = \frac{1}{2\ m}\ \delta^{\alpha\ \gamma}\ \{\ p_\alpha - e\ A_\alpha\ (q\ ,\ t)\}\ \{\ p_\gamma - e\ A_\gamma\ (q\ ,\ t)\} + e\ \varphi\ (q\ ,\ t)\ ;$$

$$H = \frac{1}{2\ m}\ (\vec{p} - e\ \vec{A})^2 + e\ \varphi\ (q\ ,\ t)$$

$$\Rightarrow\ \ m\ \ddot{\vec{q}} = \vec{F} = e\ (\vec{v} \times \vec{B} + \vec{E})\ ,$$

$$\vec{E} = -\vec{\nabla}\ \varphi - \frac{\partial\ \vec{A}}{\partial\ t}\ ,$$

$$\text{curl}\ \vec{A} = \vec{B}$$

7.1 A Brief Review of Special Relativity

quantum mechanics: $h \to 0$ $\to$ classical mechanics

relativity: $c \to \infty$ $\to$ classical mechanics

Einstein's special theory of relativity: compares the way a physical system is described by different observers moving at *constant velocity* with respect to each other.

Galilean transformation ($V \ll c$): One observer sees a certain event $\vec{x}\ ,\ t$; second observer $\vec{V}\ :\ \vec{x'}\ ,\ t'$;

$\vec{x'} = \vec{x} - \vec{V}\ t\ ,\ t' = t$

Lorentz transformation ($\vec{V} \| x_1$ -axis):

$$x'^1 = \gamma \left[x^1 - \frac{V\ x^0}{c} \right],\ x'^0 = \gamma \left[x^0 - \frac{V\ x'}{c} \right],\ x'^2 = x^2\ ,\ x'^3 = x^3\ ,\ x^0 \equiv c\ t\ ,\ \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$$

Introduce 4-vectors $x = (x^0\ ,\ x^1\ ,\ x^2\ ,\ x^3)$ ($c = 1$, light years for distance)

The geometry of the 4D-space-time-manifold is called **Minkowskian**.

In **Euclidean** gemoetry

$$\Delta\ s^2 = (\Delta\ x^1)^2 + (\Delta\ x^2)^2 + (\Delta\ x^3)^2$$

is invariant, where $\Delta \vec{x}$ is the separation between two points.
In Minkowskian geometry:
$$\Delta \sigma^2 \equiv g_{\mu\nu} \Delta x^\mu \Delta x^\nu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$
where $\Delta x \equiv (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3)$ is the space-time separation between two **events**. $\mu = 0, 1, 2, 3$.
Minkowskian metric $g_{\mu\nu} = 0$ if $\mu \neq \nu$; $g_{11} = g_{22} = g_{33} = -g_{00} = 1$

Special relativistic generalization of Newton's equations is
$$m \frac{d^2 x^\mu}{d \tau^2} \equiv m a^\mu = F^\mu \left(x, \frac{d x}{d \tau}, \tau \right)$$
x, a, F are four-vectors. The mass (called the **rest mass**) is a scalar. The scalar τ is called the **proper time** („Eigenzeit“): it is measured along the motion and is defined by
$$(d \tau)^2 = -g_{\mu\nu} d x^\mu d x^\nu$$
$$-(d \tau)^2 = (d x^1)^2 + (d x^2)^2 + (d x^3)^2 - d t^2 = \vec{v}^2 d t^2 - d t^2 = -(1 - \vec{v}^2) d t^2$$
$$d \tau = \frac{d t}{\gamma}, \gamma = \frac{1}{\sqrt{1 - \vec{v}^2 / c^2}}$$

4-vector generalization of the 3-vector velocity has components
$$u^\mu = \frac{d x^\mu}{d \tau}, \; g_{\mu\nu} u^\mu u^\nu = -1$$
$\Rightarrow$ velocities are limited in magnitude
$$u^0 = \gamma$$
$$u^k = \gamma v^k \equiv \gamma \frac{d x^k}{d t} \qquad k = 1, 2, 3$$
v^k is the nonrelativistic velocity

Want to find Hamiltonian of this relativistic dynamics:
$$L_R = T(\vec{v}) - U(\vec{x})$$

T, U are relativistic generalizations of the kinetic and potential energies, $\vec{v}$ and $\vec{x}$ are the 3-vectors.

The 3-velocity should satisfy the equation
$$\frac{\partial L_R}{\partial v^k} = p^k$$
(Latin indices go from 1 to 3, Greek indices from 0 to 3)
$$p^k = m u^k = m \gamma v^k = \frac{m v^k}{\sqrt{1 - \vec{v}^2}}, \; p^0 = m \gamma$$

where $\vec{v}^2 = (v^1)^2 + (v^2)^2 + (v^3)^2$. $\vec{v}$ is the 3-velocity of the particle.
$$\frac{\partial T}{\partial v^k} = \frac{m v^k}{\sqrt{1 - \vec{v}^2}}$$
$$\Rightarrow T = -m \sqrt{1 - \vec{v}^2} + C = \frac{-m}{\gamma} + C;$$

$$v \ll c \Rightarrow T \approx \frac{1}{2} m \vec{v}^2 + u$$

Relativistic Lagrangian of a conservative one-particle system is

$$L_R = -\frac{m}{\gamma} - U(\vec{x})$$

Relativistic expression for energy

$$E_R = \frac{\partial L_R}{\partial v^k} v^k - L_R = p^k v^k - L_R = p^k v^k - L_R = m\gamma\vec{v}^2 + \frac{m}{\gamma} + U(\vec{x})$$

$$= m\gamma\left(1 - \frac{1}{\gamma^2}\right) + \frac{m}{\gamma} + U(\vec{x}) = m\gamma + U(\vec{x})$$

$$\Rightarrow H_R = \frac{\partial L_R}{\partial v^k} v^k - L_R = \frac{p^k p^k}{m\gamma} + \frac{m}{\gamma} + U(\vec{x}),$$

$$\gamma = \frac{1}{\sqrt{1-\vec{v}^2}} = \sqrt{1 + \frac{\vec{v}^2}{1-\vec{v}^2}} = \sqrt{1+\gamma^2\vec{v}^2}$$

$$m\gamma = \sqrt{\vec{p}^2 + m^2} \qquad \vec{p}^2 \equiv p^k p^k$$

$\Rightarrow$

$$H_R = \sqrt{\vec{p}^2 + m^2} + U(\vec{x})$$

6.1.1 The Relativistic Kepler Problem

We take

$$U = -\frac{e^2}{r}.$$

1916 Sommerfeld obtains correct first-order expressions for the fine structure of the hydrogen spectrum.

$$H = \sqrt{\vec{y}^2 + m^2} - \frac{e^2}{r}, \; r^2 = \vec{x}\cdot\vec{x}$$

Motion is constrained to a plane $\Rightarrow$ number of freedoms is 2: r, φ

$$H = \sqrt{p_r^2 + \frac{p_\varphi^2}{r^2} + m^2} - \frac{e^2}{r}, \; p_\varphi = \frac{\partial L}{\partial\dot{\varphi}}, \; p_r = \frac{\partial L}{\partial\dot{r}}$$

φ is again cyclic $\Rightarrow$ $p_\varphi =$ conserved

$r = r(\varphi)$ instead of $r = r(t)$, $\varphi = \varphi(t)$

Hamiltonian equations of motion $\Rightarrow$

$$r = \frac{q}{1 + \varepsilon\cos[\Gamma(\varphi - \varphi_0)]}$$

where ε, Γ, q depend on e, c, p_φ and the total energy E.

$$\Gamma = \sqrt{1 - \frac{e^2}{c^2 p_\varphi^2}} \neq 1$$

$r = r(\varphi)$ is *not* a conic section, i.e. the orbit does not close. Orbit looks like a **precessing ellipse**.
$\Rightarrow$ fine structure in a hydrogen atom.

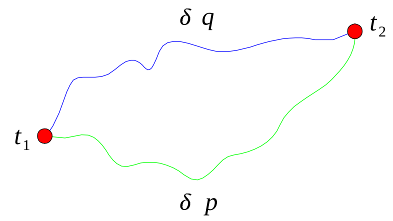
7.2 Canonical Transformations

7.2.1 Hamilton's Principle

$$\delta S[q] = \delta \int_{t_1}^{t_2} \mathrm{d}t\, L = \delta \int \mathrm{d}t\,(p_\alpha \dot{q}_\alpha - H(q, p, t)) = 0$$

$$\alpha = 1, \ldots, n \qquad S = S[q], q(t_1), q(t_2)$$

$S[q, p]$. For given boundary values
$\delta q(t_1) = \delta q(t_2) = 0$, $\delta p(t_1) = \delta p(t_2) = 0$



$$\delta S[q, p] = \int_{t_1}^{t_2} \mathrm{d}t \left[\left(\dot{q}_\alpha - \frac{\partial H}{\partial p_\alpha} \right) \delta p^\alpha - \left(\dot{p}_\alpha + \frac{\partial H}{\partial q_\alpha} \right) \delta q^\alpha \right]$$

used:

$$\int p_\alpha \delta \dot{q}_\alpha \,\mathrm{d}t = -\int \dot{p}_\alpha \delta q_\alpha \,\mathrm{d}t$$

Hamiltonian equations of motion $\Leftrightarrow \delta S = 0$
$\delta S[q, p] = 0$ **Hamilton's Principle**

Because we have $\delta q = 0$ and $\delta p = 0$ at the boundaries, we can add

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q, p, t),$$

the transformation

$$p_\alpha \dot{q}_\alpha - H \rightarrow p_\alpha \dot{q}_\alpha - H + \frac{\mathrm{d}}{\mathrm{d}t} F(q, p, t)$$

leaves the canonical equations of motion invariant.

7.2.2 Choice of Coordinates and Momenta

Can we introduce new coordinates and momenta in such a way that the canonical equations have the same form but are easier to solve? Can we find a function $H'(q, p, t)$ such that
$Q_\alpha = Q_\alpha(q_1, p, t)$, $P_\alpha = P_\alpha(q, p, t)$ (**canonical transformation**)
that do:

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

into:

$$\dot{Q}_\alpha = \frac{\partial H'}{\partial P_\alpha}, \quad \dot{P}_\alpha = -\frac{\partial H'}{\partial Q_\alpha}$$

?

$$\delta \int_{t_1}^{t_2} \mathrm{d}\, t\, (p_\alpha\, \dot{q}_\alpha - H\,(q\,,\,p\,,\,t)) = \delta \int_{t_1}^{t_2} \mathrm{d}\, t\, (P_\alpha\, \dot{Q}_\alpha - H'\,(Q\,,\,P\,,\,t)) = 0$$

$$\Rightarrow\; p_\alpha\, \dot{q}_\alpha - H = P_\alpha\, \dot{Q}_\alpha - H' + \frac{\mathrm{d}}{\mathrm{d}\, t}\, F\,(q\,,\,p\,,\,Q\,,\,P\,,\,t)$$

Function F is called **generating function** of the transformation.
F depends only on 3 sets of independent variables such as $q\,,\,P\,,\,t$ or $q\,,\,Q\,,\,t$.

Let us choose $F = F\,(q\,,\,Q\,,\,t)$:

$$\frac{\mathrm{d}\, F\,(q\,,\,Q\,,\,t)}{\mathrm{d}\, t} = \frac{\partial\, F}{\partial\, q_\alpha}\, \dot{q}^\alpha + \frac{\partial\, F}{\partial\, Q_\alpha}\, \dot{Q}_\alpha + \frac{\partial\, F}{\partial\, t} \quad \Rightarrow$$

$$p_\alpha = \frac{\partial\, F\,(q\,,\,Q\,,t)}{\partial\, q^\alpha} = p_\alpha\,(q\,,\,Q\,,\,t)$$

$$P_\alpha = -\frac{\partial\, F\,(q\,,\,Q\,,t)}{\partial\, Q^\alpha} = P_\alpha\,(q\,,\,Q\,,\,t)$$

$$H'\,(Q\,,\,P\,,\,t) = H\,(q\,,\,p\,,\,t) + \frac{\partial\, F\,(q\,,\,Q\,,\,t)}{\partial\, t}$$

Example: The Hamiltonian of an oscillator is

$$H = \frac{p^2}{2\, m} + \frac{m\,\omega^2\, q^2}{2} \Leftrightarrow \dot{q} = \frac{p}{m}\,,\; \dot{p} = -m\,\omega^2\, q$$

For the generating function

$$F\,(q\,,\,Q) = \frac{m\,\omega}{2}\, q^2 \cot Q$$

$$p = \frac{\partial\, F}{\partial\, q} = m\,\omega\, q \cot Q$$

$$P = -\frac{\partial\, F}{\partial\, Q} = \frac{m\,\omega\, q^2}{2 \sin^2 Q}$$

Can be inverted to give

$$q = \sqrt{\frac{2\, P}{m\,\omega}} \sin Q\,,\; p = \sqrt{2\, m\,\omega\, P} \cos Q$$

$$H'\,(Q\,,\,P) = H\,(q\,(Q\,,\,P)\,,\; p\,(Q\,,\,P)) = \omega\, P \cos^2 Q + \omega\, P \sin^2 Q = \omega\, P$$

The new variable Q is cyclic. The canonical equations are

$$\dot{P} = -\frac{\partial\, H'}{\partial\, Q} = 0\;,\; \dot{Q} = \frac{\partial\, H'}{\partial\, P} = \omega$$

$$\Rightarrow\; P = const = \frac{H'}{\omega} = \frac{E}{\omega}\,;\; Q\,(t) = \omega\, t + \beta\,;$$

7.3 Hamilton-Jacobi-Equation

Can we find a canonical transformation $W\,(q\,,\,Q\,,\,t)$ such that

(*) $H'\,(Q\,,\,P\,,\,t) = H\,(q\,,\,p\,,\,t) + \dfrac{\partial\, W}{\partial\, t} = 0$?

Condition leads to a PDE for W that is called the **Hamilton-Jacobi equation**. It is ideally suited for studying the relations between mechanics, optics, quantum mechanics. The equation (*) obeys:

$$p_\alpha = \frac{\partial W(q, Q, t)}{\partial q^\alpha} = p_\alpha(q, Q, t)$$
$$P_\alpha = -\frac{\partial W(q, Q, t)}{\partial Q^\alpha} = P_\alpha(q, Q, t)$$
$$H' = 0 \Rightarrow \dot{Q}_\alpha = 0, \dot{P}_\alpha = 0 \Rightarrow Q_\alpha = a_\alpha = const, P_\alpha = b_\alpha = const$$

Hamilton-Jacobi equation:

$$H\left(q_\alpha, \frac{\partial W(q, a, t)}{\partial q_1}, \dots, \frac{\partial W(q, a, t)}{\partial q_n}, t\right) + \frac{\partial W(q, a, t}{\partial t}$$

Assume we have solved this PDE and found
$W(q_1, q_2, \dots, q_n, a_1, \dots, a_n, t)$
that depends on n constants of integration a_α. This solution leads to the equation

$$p_\alpha = \frac{\partial W(q, a, t)}{\partial q^\alpha} \overset{\text{inversion}}{\rightarrow} q_\alpha = q_\alpha(a, b, t)$$
$$b_\alpha = -\frac{\partial W(q, a, t)}{\partial a^\alpha} \overset{\text{inversion}}{\rightarrow} p_\alpha = p_\alpha(a, b, t)$$

If H does not explicitly depend on time,
$W(q, a, t) = S(q, a) - E\,t$

$$H\left(q_\alpha, \frac{\partial S(q, a)}{\partial q^\alpha}\right) = E$$

Example (cone on a tilted plane (homework)):

$$H = \frac{1}{2m}\left(p_x^2 + p_z^2\right) + m\,g\,z$$
$$\frac{1}{2m}\left(\frac{\partial S(x, z)}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S(x, z)}{\partial z}\right)^2 + m\,g\,z = E$$

PDE Ansatz:

$$S(x, z) = S_1(x) + S_2(z)$$
$$\left(\frac{\mathrm{d} S_1}{\mathrm{d} x}\right)^2 + \left(\frac{\mathrm{d} S_2}{\mathrm{d} z}\right)^2 + 2\,m^2\,g\,z = 2\,m\,E \quad \text{or}$$
$$\left(\frac{\mathrm{d} S_1}{\mathrm{d} x}\right)^2 = f(z) = const = a_2$$
$$\frac{\mathrm{d} S_1}{\mathrm{d} x} = \sqrt{a_2}$$
$$\frac{\mathrm{d} S_2}{\mathrm{d} z} = \sqrt{2\,m\,E - a_2 - 2\,m^2\,g\,z}$$
$$S_1 = \sqrt{a_2}\,x$$
$$S_2 = -\frac{1}{3\,m^2\,g}\left(2\,m\,E - a_2 - 2\,m^2\,g\,z\right)^{\frac{3}{2}}$$

$$S = S(x, z, E, a_2) = \sqrt{a_2}\, x - \frac{1}{3 m^2 g} (2 m E - a_2 - 2 m^2 g z)^{\frac{3}{2}}$$

$$p_x = \frac{\partial S}{\partial x},\ p_z = \frac{\partial S}{\partial z},\ b_1 = -\frac{\partial S}{\partial E} + t,\ b_2 = -\frac{\partial S}{\partial a_2}$$

$$\Rightarrow x(t), z(t), p_x(t), p_z(t)$$

$$-\frac{1}{m g} \sqrt{2 m E - a_2 - 2 m^2 g z} = t - b_1$$

$$\Rightarrow z(t) = -\frac{g}{z} (t - b_1)^2 + \frac{2 m E - a_2}{2 m^2 g} = -\frac{g}{2} t^2 + c_1 t + c_2$$

$\Rightarrow$ parabola

8. Nonlinear Dynamics

8.1 Introduction

Irregular, unpredictable time evolution = „chaos“

Chaotic dynamical systems are deterministic.
For non-chaotic systems: slight change in initial conditions leads to differences in the phase space trajectories that grow *linearly* in time.
For chaotic system, the error grows *exponentially* in time.
$\Rightarrow$ „Sensitivity to initial conditions.“ (*1913 Poincaré*)

For chaos, the irregularity is not caused by random forces, but is part of the intrinsic dynamics of the deterministic system.

Necessary conditions for chaotic motion:

(a) the system has at least 3 independent dynamical variables

(b) the equations of motion contain a *nonlinear* term that couples at least 2 of the variables

Example: damped, sinusoidally driven pendulum of mass m (weight W) and length l:

$$m\ l^2 \frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + W\ l \sin\theta = A \cos(\omega_D\ t)$$

In dimensionless form:

$$\frac{d\theta}{dt^2} + \frac{1}{q}\frac{d\theta}{dt} + \sin\theta = g\ \cos(\omega_D\ t)$$

q is the damping (or quality) parameter, g is the forcing amplitude, ω_D is the drive frequency.

Rewrite:

$$\frac{d\omega}{dt} = -\frac{1}{q}\ \omega - \sin\theta + g\ \cos\phi$$
$$\frac{d\phi}{dt} = \omega_D$$
$$\frac{d\theta}{dt} = \omega$$

ϕ is the phase of the drive term. 3 variables (ω , θ , ϕ).

8.2 Tools for Chaotic Systems

8.2.1 Phase Space

Phase space: diagram $(\vec{x}, \vec{v})$ or $(\vec{x}, \vec{p})$

Undamped pendulum in small amplitude approximation: $\sin\theta \approx \theta$

$$\frac{d^2}{dt^2} + \theta = 0\ ;\ \ \omega \equiv \frac{d\theta}{dt} \Rightarrow$$
$$\frac{d\omega}{dt} = -\theta\ ,\ \frac{d\theta}{dt} = \omega \Rightarrow \theta = a\ \cos t\ ,\ \omega = a\ \sin t\ .$$

In a *deterministic* system, trajectories have the following properties:

(a) Orbits do not cross each other.

(b) Conservative deterministic systems *preserve areas* in phase space, i.e. all points in a given area of phase space move in such a way that these points occupy the same area at all times.

The property of volume/area preservation leads to a classification of a dynamical system into 2 categories:

- conservative (linearized, undamped)
- dissipative (linearized, damped)

depending on whether the phase volumes *stay constant* or *contract*, respectively

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + \theta = 0 \rightarrow \omega = \theta = 0$$

Attractor: Point in phase space: a finite set of initial coordinates (θ, ω) converges to it. The set of curves that divide one basin of attraction from another is called **separatrix**.

8.2.2 Poincaré Section

Fully nonlinear, driven, and damped oscillation. For moderately driven pendulum system, the resulting closed orbit is an attractor, it is called a **limit cycle**.

A Poincaré section is constructed by viewing the phase space diagram *stroboscopically* in such a way that the motion is observed periodically. For a driven pendulum, the strobe period is the period of the forcing ω_D.

8.3 Visualization of the Pendulum's Dynamics

$$\frac{d\omega}{dt} = -\frac{\omega}{q} - \sin\theta + g\cos\phi$$

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\phi}{dt} = \omega_D$$

8.3.1 Sensitivity to Initial Conditions

Observe the phase space evolution of a block of pendulum states.

8.3.2 Phase Diagrams: Chaos and Self-Similarity

- Drive force amplitudes g will be varied.
- $\omega_D = \frac{2}{3}$, $q = 2$

  3-D space (θ, ω, ϕ), 2-D space, Poincaré sections

$g = 1.5$: Poincaré section shows a chaotic attractor, called **strange attractor**. The fine structure, when magnified, resembles the gross structure. This property is called **self-similarity**. One can characterize these objects by non-integer or **fractal dimension**:
$d > 1$ but $d < 2$, the attractor is more space-filling than a line but less than an area.

8.3.3 Bifurcation Diagrams

View the dynamics more globally over a range of parameter values: **bifurcation diagrams**. In

dynamics, a change in the number of solutions to a differential equation if a parameter is varied is called bifurcation.
Pendulum: examine a graph of ω at a fixed phase in the drive cycle versus the drive amplitude.

Whenever another stable stationary solution pops up, one has the effect of **period doubling**. System cycles between two values of ω for each set of initial conditions.

Mechanical oscillator possesses 2 stable oscillation frequencies; the system can be tuned to either of them by appropriate initial conditions: *mechanical flip-flop*.

8.4 Towards an Understanding of Chaos

$$x_{n+1} = f(\mu, x_n)$$

x_n n-th value of $x \in (0, 1)$; μ : parameter; $f : (0, 1) \to (0, 1)$ map

8.4.1 The Logistic Map

$$x_{n+1} = f(x_n) = \mu\, x_n (1 - x_n),\ x \in [0, 1]$$

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \mu\, x (1 - x) \qquad \textit{Verhulst, 1845}$$

The fixed point is stable for all initial conditions when $\mu = 2$. Remains true for all μ where $|f'(x)| < 1$ at the intersection of f and $x_{n+1} = x_n$. As soon as $|f'(x)| > 1$ at the intersection, the fixed point oscillates. The passage to chaos through a sequence of period doublings is one important feature of the logistic map. The period doubling mechanism is one **route to chaos**.
The ratio of spacings between consecutive values of μ at the bifurcations approaches an universal constant, called **Feigenbaum number**.

$$\lim_{k \to \infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \delta = 4.669201 \ldots$$

8.4.2 Ljapunov Exponent

= Quantitative measure of the sensitive dependence upon initial conditions.

Initial states, x and $x + \varepsilon$, then after n iterations their divergence may be characterized by
$$\varepsilon(n) \approx \varepsilon\, e^{\lambda n}$$
where Lyapunov exponent λ gives the *average rate of divergence.*
If $\lambda < 0$ $\Rightarrow$ evolution is not chaotic.
If $\lambda > 0$ $\Rightarrow$ nearby trajectories diverge $\Rightarrow$ chaotic.

$$x_{n+1} = f(x_n)$$

After n-th step:
$$f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0) \approx \varepsilon\, e^{n \lambda} \text{ or}$$
$$\ln\left[\frac{f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0)}{\varepsilon}\right] \approx n\, \lambda$$

$f^{(n)}(x) = f(f(f(\ldots f(x))))$ is the value after n-th iteration. For small ε,
$$\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \ln \left|\frac{\mathrm{d}\, f^{(n)}(x_0)}{\mathrm{d}\, x_0}\right|$$

Apply chain rule for the second iteration:

$$\frac{d}{d\,x} f^{(2)}(x)\Big|_{x_0} = \frac{d}{d\,x} f(f(x))\Big|_{x_0} = f'(f(x_0)) \cdot f'(x_0) = f'(x_1) \cdot f'(x_0)$$

$$x_1 = f(x_0)$$

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{d\, f^{(n)}(x_n)}{d\, x_0} \right| = \lim_{n\to\infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{n=0}^{n-1} \ln |f'(x_i)|$$

The Ljapunov exponent gives the stretching rate per iteration, averaged over the trajectory.

8.4.3 Intermittency Route to Chaos

By **intermittency** we mean the occurance of a signal that alternates randomly between long, regular, periodic phases (so-called **intermissions**) and relatively short irregular bursts.

$x_{n+1} = f(x_n)$ if map approaches but does not cross the line $x_{n+1} = x_n$ and runs nearly tangential, one gets intermittency:
Celestial mechanics: Asteroids between Mars and Jupiter: maintain seemingly stable, almost circular orbits for years, suddenly develop large eccentricities $\Rightarrow$ shooting stars.

8.4.4 Phase Locking

Phase locking is said to occur when the ratio of the frequency of a pendulum to that of the forcing becomes locked to a ratio $n/m \ (n, m \in \mathbb{N})$, over some finite domain of parameter values.

Standard Map

$$\Theta_{n+1} = \left[\Theta_n + \Omega - \frac{K}{2\,\pi} \sin(2\,\pi\,\Theta_n) \right] mod\ 1$$

There are 2 parameters (Ω, K) and $\Theta_n \in (0, 1)$. The modulo function guarantees periodic boundary conditions. Ω is rotation frequency of „winding number" in the absence of nonlinearity ($K = 0$). K strength of nonlinear coupling of the oscillator to the forcing.

$K = 0$:
$\Theta_{n+1} = \Theta_n + \Omega \ mod\ 1$;

$\Omega = 0.4$:	0.3	0.7	0.1	0.5	0.9	0.3

After 5 iterations return to initial value; 2 revolutions; $\Rightarrow W = 2/5 = 0.4 = \Omega$.

Whenever Ω is rational, the map is periodic. When nonlinear term is added, mode locking occurs: the motion becomes periodic even when Ω is irrational.

8.5 Characterization of Chaotic Attractors: Fractals

Dimension of a geometrical object: **Hausdorff-dimension** or **capacity-dimension**.

Consider a curve of total length L . Assume that we can completely cover this line by $N(\varepsilon)$ one-dimensional boxes of size ε on a side.

$$N(\varepsilon)\cdot\varepsilon = L \; ; \; N(\varepsilon) = L\frac{1}{\varepsilon} \; ;$$

2-D square of side L can be covered by

$$N(\varepsilon) = L^2\left(\frac{1}{\varepsilon}\right)^2$$

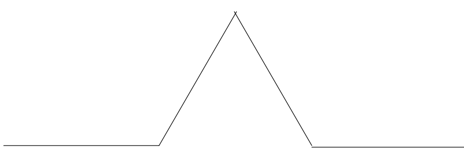
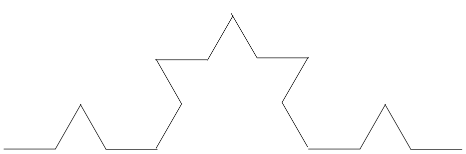
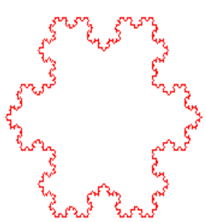
boxes.

For 3-D cube: Exponents are 3.

$$N(\varepsilon) = L^d\left(\frac{1}{\varepsilon}\right)^d \Rightarrow d = \frac{\ln N(\varepsilon)}{\ln L + \ln(1/\varepsilon)}$$

$$d_{Hausdorff} = \lim_{\varepsilon\to 0}\frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$$

Koch curve:

	N	ε
	1	1
	4	1/3
	16	1/9
	4^n	$\left(\frac{1}{3}\right)^n$

$$d_H = \frac{n\ln 4}{n\ln 3} = \frac{\ln 4}{\ln 3} = 1.26$$

9. Basic Concepts of Electrodynamics

9.1 Electrostatics

Static charge distribution $\varrho\left(\vec{r}\right)$. Force that is felt by a test charge ($+e$) due to $\varrho\left(\vec{r}\right)$:
Force $\varrho = e\,\vec{E}\left(\vec{x}\right)$

$$\vec{E}\left(\vec{x}\right) = \frac{1}{4\,\pi\,\varepsilon_0} \int \varrho\left(\vec{x'}\right) \frac{\left(\vec{x} - \vec{x'}\right)}{\left|\vec{x} - \vec{x'}\right|^3}\,\mathrm{d}^3\,x'$$

ε_0 = vacuum permittivity or dielectric constant of vacuum

$$\varepsilon_0\,\vec{\nabla}\cdot\vec{E} = \varrho$$

Gauss's Law

$$E\left(\vec{x}\right) = -\frac{1}{4\,\pi\,\varepsilon_0}\,\vec{\nabla} \int \frac{\varrho\left(\vec{x'}\right)}{\left|\vec{x} - \vec{x'}\right|}\,\mathrm{d}^3\,x' \equiv -\vec{\nabla}\,\phi\left(\vec{x}\right)$$

curl grad $\equiv 0$:

$$\vec{\nabla}\times\vec{E} = 0$$

$$\nabla^2\,\phi - \frac{\varrho}{\varepsilon_0}$$

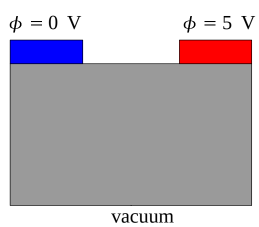
Poisson equation

PDE for $\phi\left(\vec{x}\right)$:

(a) ϱ

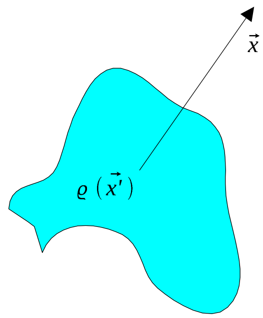
(b)boundary conditions for $\phi\left(\vec{x}\right)$ on a closed surface
2 types of boundary conditions that occur frequently:

- Dirichlet boundary condition: ϕ = preset value



- von-Neumann boundary condition: $E_n = -\nabla_n\,\phi$ (= normal derivative of ϕ) is specified on a closed surface: $\nabla_n\,\phi = 0$

9.1.1 Multipole Expansion



$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\varrho(\vec{x'})}{|\vec{x}-\vec{x'}|}\,\mathrm{d}^3x'$$

Taylor expansion of $1/|\vec{x}-\vec{x'}|$ in terms of powers of $\vec{x'}$:

$$\frac{\partial}{\partial x'_i}\frac{1}{|\vec{x}-\vec{x'}|}\Bigg|_{\vec{x'}=0} = -\frac{\partial}{\partial x_i}\frac{1}{|\vec{x}-\vec{x'}|}\Bigg|_{\vec{x'}=0} = -\frac{\partial}{\partial x_i}\frac{1}{r}\,;\; r\equiv|\vec{x}|\,;$$

$$\frac{1}{|\vec{x}-\vec{x'}|} = \frac{1}{r} - \sum_{i=1}^{3} x'_i \frac{\partial}{\partial x_i}\frac{1}{r}$$

$$f(\vec{x}-\vec{x'}) = f(\vec{x}) + \sum_{i=1}^{3} x'_i \frac{\partial f}{\partial x'_i} + \ldots + \frac{1}{2}\sum_{i,j=1}^{3} x'_i x'_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{1}{r} + \ldots$$

$$4\pi\varepsilon_0\,\phi(\vec{x}) = \frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j=1}^{3} Q_{ij}\frac{x_i x_j}{r^5} + \ldots$$

$q = \int \mathrm{d}^3x'\,\varrho(\vec{x'})$ is total charge
$\vec{p} = \int \mathrm{d}^3x'\,\vec{x'}\,\varrho(\vec{x'})$ dipole moment
$Q_{ij} = \int \mathrm{d}^3x'\,(3x'_i x'_j - r'^2\delta_{ij})\,\varrho(\vec{x'})$ quadrupole moment

$\Delta\frac{1}{r} = 0$ for $r \neq 0$

9.2 Magnetostatics

$\vec{\nabla}\cdot\vec{B} = 0$ no free magnetic charges, no monopoles

Electric current density: $\vec{J}$ = in units of positive charge crossing unit area per unit time.
Direction of motion $\parallel \vec{J}$
Units: Amperes per square meter
Current density is confined to a wire $\Rightarrow$ current $= J\cdot A$.

Microscopically: point charges at $\vec{x}_i$ with velocities $\vec{v}_i$:

$$\varrho(\vec{x},t) = \sum_i q_i\,\delta(\vec{x}-\vec{x}_i(t))$$

$$\vec{J}(\vec{x},t) = \sum_i q_i\,\vec{v}_i(t)\,\delta(\vec{x}-\vec{x}_i(t))$$

δ : Dirac's delta function (is a functional):
$\int \mathrm{d}x\, f(x)\,\delta(x-x_0) = f(x_0)$

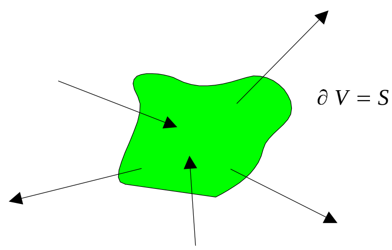
Distribution:

$$\lim_{\varepsilon \to 0} \int \mathrm{d}\, x\, f(x) \frac{\varepsilon}{(x - x_0)^2 + \varepsilon^2} = f(x_0)$$

Conservation of charge demands relation between $\varrho \leftrightarrow \vec{J}$ via continuity equation

$$\vec{J}(\vec{x}, t)\, \Delta V = \int_{\Delta V} \mathrm{d}^3 x\, q(\vec{x})\, \vec{v}(\vec{x}, t)$$

$$\frac{\mathrm{d}}{\mathrm{d}\, t} = \int_V \mathrm{d}^3 x'\, \varrho(\vec{x'}, t) + \oint_S \mathrm{d}\, \vec{S} \cdot \vec{J}(\vec{x'}, t) = 0$$



Gauss's theorem:

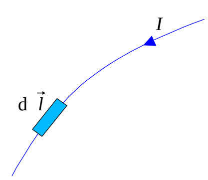
$$\int_V \mathrm{d}^3 x' \left(\frac{\partial\, \varrho(\vec{x'}, t)}{\partial\, t} + \mathrm{div}\, \vec{J}(\vec{x'}, t) \right) = 0$$

$$\frac{\partial\, \varrho}{\partial\, t} + \vec{\nabla} \cdot \vec{J} = 0$$

Magnetostatics: steady-state phenomene $\Rightarrow \vec{\nabla} \cdot \vec{J} = 0$

Consider a filamentary wire which carries a constant current I. In the presence of other conductors (e.g. current-carrying wires) that wire feels a force. The elemental force $\mathrm{d}\, \vec{F}$ experienced by a current element $\mathrm{d}\, \vec{l}$ of the wire obeys

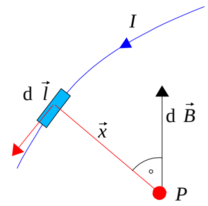
$$\mathrm{d}\, F \propto I\, ,\ \mathrm{d}\, F \propto |\mathrm{d}\, \vec{l}|\, ,\ \mathrm{d}\, \vec{F} \perp \mathrm{d}\, \vec{l}$$



$$\mathrm{d}\, \vec{F}(\vec{x}) = I\, \mathrm{d}\, \vec{l} \times \vec{B}(\vec{x})$$

magnetic induction $\vec{B}$ or **magnetic flux density** $\vec{B}$

We now determine the magnitude of $\vec{B}$ generated by a current-carrying wire: Consider again a filamentary wire.



The elemental flux density $\mathrm{d}\,\vec{B}$ generated by a current element $\mathrm{d}\,\vec{l}$ at an observation point P is

$$\mathrm{d}\,\vec{B} = \frac{\mu_0}{4\,\pi}\,I\,\frac{\mathrm{d}\,\vec{l}\times\vec{x}}{|\vec{x}|^3}$$

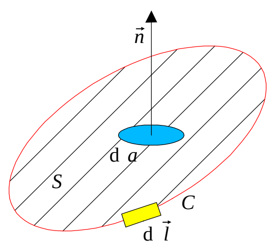
$\mu_0 = (\varepsilon_0\,c^2)^{-1}$ **vacuum permeability**

Biot-Savart-Law

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\,\pi}\int \vec{J}(\vec{x}')\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,\mathrm{d}^3x' = \frac{\mu_0}{4\,\pi}\,\vec{\nabla}\times\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x'$$

$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$ (corresponding to $\vec{\nabla}\times\vec{E} = 0$)

$$\vec{\nabla}\times\vec{B}(\vec{x}) = \mu_0\,\vec{J}(\vec{x})$$



$$\int_S \vec{\nabla}\times\vec{B}\cdot\vec{n}\,\mathrm{d}\,a = \mu_0\int_S \vec{J}\cdot\vec{n}\,\mathrm{d}\,a$$

$$\oint_C \vec{B}\cdot\mathrm{d}\,\vec{l} = \mu_0\int_S \vec{J}\cdot\vec{n}\,\mathrm{d}\,a = \mu_0\,I$$

I is the total current passing through the closed curve C . Ampere's law.

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\,\pi}\int\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,\mathrm{d}^3x' + \nabla\,\psi(\vec{x})$$

$\vec{A}\rightarrow\vec{A}+\nabla\,\psi$ **Gauge transformation**

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\,\vec{J}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\,\vec{A} = \mu_0\,\vec{J}$$

$\vec{\nabla}\cdot\vec{A} = 0$; (but $\vec{\nabla}\cdot\nabla\,\psi \neq 0$ for arbitrary ψ)

$\Rightarrow$ **Coulomb gauge** (fixes ψ)

$$\vec{\nabla}^2\,\vec{A}(\vec{x}) = -\mu_0\,\vec{J}$$

Magnetic field $\vec{H}$: $\vec{B} = \mu_0\,\vec{H}$

$\Rightarrow$ Ampere's law: $\vec{\nabla} \times \vec{H} = \vec{J}$

Dielectric displacement: $\vec{D} = \varepsilon_0 \vec{E}$

$\Rightarrow$ $\vec{\nabla} \cdot \vec{D} = \varrho$ (div $\vec{D} = \varrho$)

Force always contains $\vec{E}$ and $\vec{B}$ (H and D are only auxiliary elements)

9.3 Time-Varying Field

Time-dependent magnetic fields induce electric fields and vice versa $\Rightarrow$ **electromagnetic fields**

9.3.1 Faraday's Law

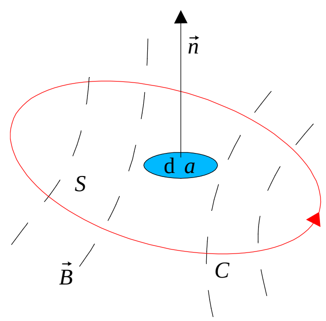
Faraday 1831 observed that a transient current is induced in a circuit if

a) the steady current flowing in an adjacent circuit is turned on or off

b) the adjacent circuit with a steady current flowing is moved relative to the first circuit

c) a permanent magnet is thrust into or out of the circuit

Faraday interpreted (correctly) the transient current flow as being due to a changing magnetic flux linked by the circuit. The changing flux induces an electric field around the circuit, the line integral of which is called **electromotive force** $\tilde{E}$:



The **magnetic flux** passing through the circuit is

$$F = \int_S \vec{B} \cdot \vec{n} \, \mathrm{d} \, a$$

The electromotive force around the circuit is

$$\tilde{E} = \oint_C \vec{E}' \cdot \mathrm{d} \, \vec{l}$$

$\vec{E}'$ is the electric field at the element $\mathrm{d} \, \vec{l}$ of the circuit measured in the rest frame of the circuit.

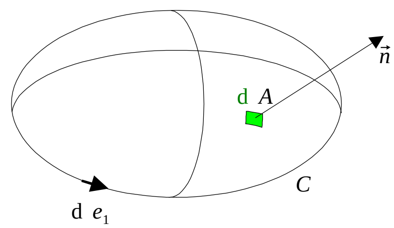
Faraday: $\tilde{E} = -\dfrac{\mathrm{d} \, F}{\mathrm{d} \, t}$

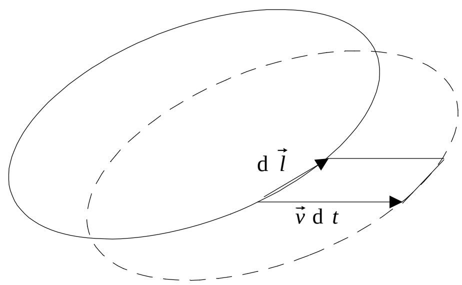
The **induced electromotive force** around the circuit is proportional to the time rate of change of magnetic flux linking the circuit.
The sign is specified by **Lenz's law**: induced current (and accompanying magnetic flux) is in such a direction as to oppose the change of flux through the circuit.

Experimentally verified that the same current is induced in a circuit whether it is moved while the circuit through which current is flowing is stationary or it is held fixed while the current-carrying circuit is moved in the same relative manner.

We can write Faraday's law more generally as

$$\oint_C \vec{E}' \cdot \mathrm{d}\,\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}\,t} \int_S \vec{B} \cdot \underbrace{\vec{n}\,\mathrm{d}\,a}_{\mathrm{d}\,\vec{a}}$$





Let us move the circuit C with velocity $\vec{v}$.

The area of the circuit changes by $\mathrm{d}\,\vec{a} = \vec{v}\,\mathrm{d}\,t \times \mathrm{d}\,\vec{l}\,;\quad \mathrm{d}\,\vec{a} = \vec{n}\,\mathrm{d}\,a$.

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \int_S \vec{B} \cdot \vec{n}\,\mathrm{d}\,a = \int_S \frac{\partial\,\vec{B}}{\partial\,t} \cdot \vec{n}\,\mathrm{d}\,a + \int_S \vec{B} \cdot \frac{\partial\,\vec{a}}{\partial\,t} = \int \frac{\partial\,\vec{B}}{\partial\,t} \cdot \vec{n}\,\mathrm{d}\,a + \oint_C \underbrace{\vec{B} \cdot (\vec{v} \times \mathrm{d}\,\vec{l})}_{= \mathrm{d}\,\vec{l} \cdot (\vec{B} \times \vec{v})}$$

$$\oint_C \left[\vec{E}' - \vec{v} \times \vec{B}\right] \cdot \mathrm{d}\,\vec{l} = -\int_S \frac{\partial\,\vec{B}}{\partial\,t} \cdot \vec{n}\,\mathrm{d}\,a$$

In an inertial frame: Let us consider S and C fixed in space.

$$\oint_C \vec{E} \cdot \mathrm{d}\,\vec{l} = -\int_S \frac{\partial\,\vec{B}}{\partial\,t} \cdot \vec{n}\,\mathrm{d}\,a$$

$\vec{E}$ is electric field in inertial system. Galilean invariance implies $\Rightarrow$ $\vec{E}'$ in the moving system,

$$\underset{\mathrm{BS}}{\vec{E}'} = \underset{\mathrm{IS}}{\vec{E}} + \vec{v} \times \vec{B}$$

Faraday's law in differential form: $\vec{E}$, $\vec{B}$ are definded in *same* frame.

$$\int_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial\,\vec{B}}{\partial\,t}\right) \cdot \vec{n}\,\mathrm{d}\,a = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial\,\vec{B}}{\partial\,t} = 0$$

9.3.2 Maxwell's Displacement Current and Maxwell's Equations

Ampere's law: $\vec{\nabla} \times \vec{H} = \vec{J}$, for steady state $\vec{\nabla} \cdot \vec{J} = 0$.

$$\vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0 \, ;$$
$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \varrho}{\partial t} = 0$$

Coulomb's law: $\vec{\nabla} \cdot \vec{D} = \varrho$ or $\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \varrho$

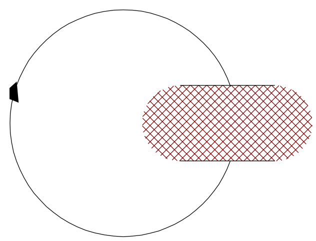
$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \varrho}{\partial t} = \vec{\nabla} \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0$$

Maxwell 1865:

$$\vec{J} \rightarrow \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Ampere's law:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$



$\Rightarrow$ Electromagnetic radiation!

Light is an electromagnetic wave phenomenon and these waves of all frequencies can be produced.

$$\vec{\nabla} \cdot \vec{D} = \varrho \quad \text{(Gauss's law)}$$
$$\vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t} = \vec{J} \quad \text{(Ampere's law)}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{(no magnetic monopoles)}$$
$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \text{(Faraday + Lenz)}$$

9.4 Wave Propagation: Plane Electromagnetic Waves

No source, infinite space

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = 0 \qquad / \; \vec{\nabla} \times$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad / \; \vec{\nabla} \times$$

$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{B})}_{\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}-\nabla^2\vec{B}}-\frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})=0$$

$$-\nabla^2\vec{B}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}=0$$

$$\Rightarrow \nabla^2 u-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0 \qquad u=E_i\,,\,B_i$$

$$u=e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

Wave vector $\vec{k}$ and frequency ω

$$-k^2+\frac{\omega^2}{c^2}=0$$

$$k=\frac{\omega}{c}=\frac{2\pi}{\lambda}$$

If we consider waves propagating in the x-direction, fundamental solution:

$$u(x,t)=A\,e^{ikx-i\omega t}+B\,e^{-ikx-i\omega t}=A\,e^{ik(x-ct)}+B\,e^{-ik(x+ct)}$$

The general solution of the wave equation is

$$u(x,t)=f(x-vt)+g(x+vt)$$

$\Rightarrow$ The equation represents traveling waves; traveling to the right and the left with velocities of propagation $v=c$.

$v=\omega/k$ is called the **phase velocity** of the wave. The **wave length** is $\lambda=(2\pi)/k$.

Convention: Physical electric and magnetic fields by taking real parts of complex quantities, plane wave fields are

$$\vec{E}(\vec{x},t)=\vec{\varepsilon}_1 E_0\,e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

$$\vec{B}(\vec{x},t)=\vec{\varepsilon}_2 B_0\,e^{i\vec{k}\cdot\vec{x}-i\omega t}$$

E_0 and B_0 are complex amplitudes which are constant in space and time.
$\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$ are constant real unit vectors.

$$\vec{\nabla}\cdot\vec{E}=0 \quad\Rightarrow\quad \vec{\varepsilon}_1\cdot\vec{k}=0$$

$$\vec{\nabla}\cdot\vec{B}=0 \quad\Rightarrow\quad \vec{\varepsilon}_2\cdot\vec{k}=0$$

Direction of propagation is $\vec{k}$ $\Rightarrow$ **transverse waves**!

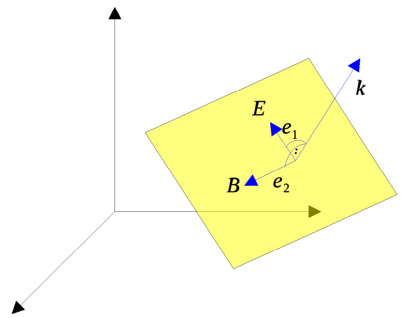
$$\vec{\nabla}\times\vec{E}+\dot{B}=0 \quad\Rightarrow\quad i\left[(\vec{k}\times\vec{\varepsilon}_1)E_0-\omega\,\vec{\varepsilon}_2 B_0\right]e^{i\vec{k}\cdot\vec{x}-i\omega t}=0$$

$\Rightarrow$ solution:

$$\vec{\varepsilon}_2=\frac{\vec{k}\times\vec{\varepsilon}_1}{k}$$

$$B_0=\frac{1}{c}E_0$$

$(\vec{\varepsilon}_1,\vec{\varepsilon}_2,\vec{k})$ form a set of orthogonal vectors and $\vec{E}$, $\vec{B}$ are in phase and possess a constant ratio.



⇒ Waves in vacuum (no sources) are transverse waves propagating in the direction of $\vec{k}$. Since $\vec{E} \parallel$ fixed direction $\vec{\varepsilon}_1$, we call such a wave **linearly polarized** with polarization vector $\vec{\varepsilon}_1$.

9.4.1 Superposition of Waves

Since ME are *linear* PDEs, any superposition of plane wave solutions is a solution as well. Waves in x-direction:

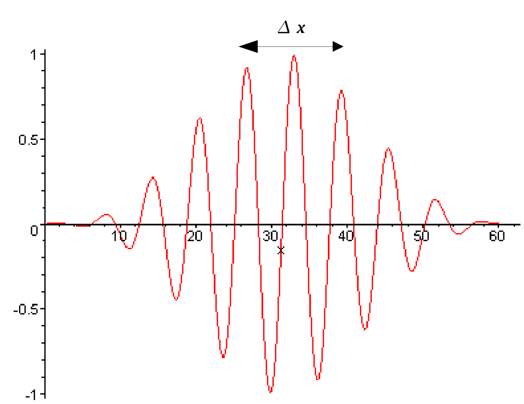
$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i k x - i \omega t}\, \mathrm{d}k$$

The amplitude $A(k)$ describes the properties of the linear superposition of the different waves. From Fourier analysis,

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0)\, e^{-i k x}\, \mathrm{d}x$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{i k x}\, \mathrm{d}k, \quad A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i k x} f(x)\, \mathrm{d}x$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i (k - k') x}\, \mathrm{d}x = \delta(k - k'), \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k (x - x')}\, \mathrm{d}k = \delta(x - x')$$



$$\Delta x \cdot \Delta k \geq \frac{1}{2}$$

In vacuum, a pulse or wave packet propagates in such a way that the packet does not change its shape. In a medium, however,

$$\omega(k) = \frac{c\, k}{n(k)},$$

so that

$$v=\frac{\omega}{k}=\frac{c}{n(k)}$$

depends on k or on the wavelength $\lambda=(2\pi)/k$. Then, wave packet will spread with time as it travels along, because the waves of different wavelength propagate at different speeds. This phenomenon is called **dispersion**.