Elastische Wellen in kubischen Medien

Lösung von Michael Wack

Elastische Energiedichte eines kubischen Kristalls

$$u = \frac{1}{2}c_{11}\left(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2\right) + c_{12}\left(\epsilon_{11}\epsilon_{22} + \epsilon_{22}\epsilon_{33} + \epsilon_{33}\epsilon_{11}\right) + 2c_{44}\left(\epsilon_{12}\epsilon_{21} + \epsilon_{23}\epsilon_{32} + \epsilon_{31}\epsilon_{13}\right)$$

a) Spannungs-Dehnungs-Beziehungen

$$\sigma_{ii} = \frac{\partial u}{\partial \epsilon_{ii}} = c_{11}\epsilon_{ii} + c_{12}(\epsilon_{jj} + \epsilon_{kk}) = c_{12}(\epsilon_{ii} + \epsilon_{jj} + \epsilon_{kk}) - c_{12}\epsilon_{ii} + c_{11}\epsilon_{ii} = c_{12}\epsilon_I + (c_{11} - c_{12})\epsilon_{ii}$$

$$\forall\ i \neq j \neq k$$

$$\sigma_{ij} = \frac{\partial u}{\partial \epsilon_{ij}} = 2c_{44}\epsilon_{ji} = 2c_{44}\epsilon_{ij} \qquad \forall\ i \neq j$$

b) Umformung der Bewegungsgleichung

Gleichgewichtsbedingung : $\dfrac{\partial \sigma_{\mathrm{ij}}}{\partial \mathrm{x_i}} - \rho \ddot{\mathrm{u}}_{\mathrm{j}} = 0 \qquad \forall\ \mathrm{j} = 1, 2, 3$

$$j = 1 \quad : \quad \frac{\partial \sigma_{11}}{\partial \mathrm{x}_1} + \frac{\partial \sigma_{21}}{\partial \mathrm{x}_2} + \frac{\partial \sigma_{31}}{\partial \mathrm{x}_3} - \rho \ddot{\mathrm{u}}_1 = 0$$

$$c_{11}\frac{\partial \epsilon_{11}}{\partial x_1} + c_{12}\frac{\partial \epsilon_{22}}{\partial x_1} + c_{12}\frac{\partial \epsilon_{33}}{\partial x_1} + 2c_{44}\frac{\partial \epsilon_{21}}{\partial x_2} + 2c_{44}\frac{\partial \epsilon_{31}}{\partial x_3} - \rho \ddot{u}_1 = 0$$

mit $\quad \epsilon_{11} = \dfrac{\partial \mathrm{u}_1}{\partial \mathrm{x}_1}, \quad \epsilon_{22} = \dfrac{\partial \mathrm{u}_2}{\partial \mathrm{x}_2}, \quad \epsilon_{33} = \dfrac{\partial \mathrm{u}_3}{\partial \mathrm{x}_3}, \quad \epsilon_{21} = \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_2}{\partial \mathrm{x}_1} + \dfrac{\partial \mathrm{u}_1}{\partial \mathrm{x}_2}\right), \quad \epsilon_{31} = \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_3}{\partial \mathrm{x}_1} + \dfrac{\partial \mathrm{u}_1}{\partial \mathrm{x}_3}\right)$

$$c_{11}\frac{\partial^2 u_1}{\partial {x_1}^2} + c_{12}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + c_{12}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + c_{44}\left(\frac{\partial^2 u_2}{\partial x_2 \partial x_1} + \frac{\partial^2 u_1}{\partial {x_2}^2} + \frac{\partial^2 u_3}{\partial x_3 \partial x_1} + \frac{\partial^2 u_1}{\partial {x_3}^2}\right) - \rho \ddot{u}_1 = 0$$

$$c_{44}\left(\Delta u_1 - \frac{\partial^2 u_1}{\partial {x_1}^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1} + \frac{\partial^2 u_3}{\partial x_3 \partial x_1}\right) + c_{12}\left(\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \frac{\partial^2 u_1}{\partial {x_1}^2} - \frac{\partial^2 u_1}{\partial {x_1}^2}\right) + c_{11}\frac{\partial^2 u_1}{\partial {x_1}^2} - \rho \ddot{u}_1 = 0$$

$$c_{44}\Delta u_1 + (c_{12} + c_{44})\frac{\partial}{\partial x_1}\mathrm{div}\ \vec{\mathrm{u}} + (\mathrm{c}_{11} - \mathrm{c}_{12} - 2\mathrm{c}_{44})\frac{\partial^2 \mathrm{u}_1}{\partial {\mathrm{x}_1}^2} - \rho \ddot{\mathrm{u}}_1 = 0$$

$$j = 2, 3 \ \text{ analog } \ \Rightarrow \ \mathrm{c}_{44}\Delta \mathrm{u_j} + (\mathrm{c}_{12} + \mathrm{c}_{44})\frac{\partial}{\partial \mathrm{x_j}}\mathrm{div}\ \vec{\mathrm{u}} + (\mathrm{c}_{11} - \mathrm{c}_{12} - 2\mathrm{c}_{44})\frac{\partial^2 \mathrm{u_j}}{\partial {\mathrm{x_j}}^2} - \rho \ddot{\mathrm{u}}_{\mathrm{j}} = 0$$

c) Ebene Welle als Lösung

$$\vec{u} = \vec{e}\ \exp\ \mathrm{i}\ \left(\vec{\mathrm{q}} \cdot \vec{\mathrm{r}} - \omega \mathrm{t}\right)$$

Einsetzen von $\vec{\mathrm{u}}$ in die Bewegungsgleichung aus Aufgabe b) :

$$u_j = e_j\ \exp\ \mathrm{i}\ \left(\mathrm{q}_1\mathrm{x}_1 + \mathrm{q}_2\mathrm{x}_2 + \mathrm{q}_3\mathrm{x}_3 - \omega \mathrm{t}\right)$$

$$\Delta u_j = \frac{\partial^2 u_j}{\partial {x_1}^2} + \frac{\partial^2 u_j}{\partial {x_2}^2} + \frac{\partial^2 u_j}{\partial {x_3}^2} = -u_j \left(q_1^2 + q_2^2 + q_3^2\right) = -u_j q^2$$

$$\frac{\partial}{\partial x_j} \mathrm{div}\ \vec{\mathrm{u}} = \frac{\partial}{\partial \mathrm{x_j}} \left(\frac{\partial \mathrm{u_1}}{\partial \mathrm{x_1}} + \frac{\partial \mathrm{u_2}}{\partial \mathrm{x_2}} + \frac{\partial \mathrm{u_3}}{\partial \mathrm{x_3}}\right) = \frac{\partial}{\partial \mathrm{x_j}}\ \mathrm{i}\left(\mathrm{u_1 q_1 + u_2 q_2 + u_3 q_3}\right) = -\mathrm{q_j}\left(\vec{\mathrm{u}} \cdot \vec{\mathrm{q}}\right)$$

$$\frac{\partial^2 u_j}{\partial x_j^2} = -q_j^2 u_j$$

$$\ddot{u}_j = -\omega^2 u_j$$

$$-c_{44} u_j q^2 - (c_{12} + c_{44}) q_j \left(\vec{u} \cdot \vec{q}\right) - (c_{11} - c_{12} - 2c_{44}) u_j q_j^2 + \rho u_j \omega^2 = 0 \qquad / \cdot \frac{-1}{\exp(\ldots)}$$

$$c_{44} q^2 e_j + (c_{12} + c_{44}) q_j \left(\vec{q} \cdot \vec{e}\right) + (c_{11} - c_{12} - 2c_{44}) q_j^2 e_j + \rho \omega^2 e_j = 0 \qquad \forall j = 1, 2, 3$$

d) mögliche Wellen in verschiedenen Ausbreitungsrichtungen

$$a = c_{44}$$

$$b = c_{12} + c_{44}$$

$$c = c_{11} - c_{12} - 2c_{44}$$

$$\begin{pmatrix} aq^2 + bq_1^2 + cq_1^2 - \rho\omega^2 & bq_1 q_2 & bq_1 q_3 \\ bq_2 q_1 & aq^2 + bq_2^2 + cq_2^2 - \rho\omega^2 & bq_2 q_3 \\ bq_3 q_1 & bq_3 q_2 & aq^2 + bq_3^2 + cq_3^2 - \rho\omega^2 \end{pmatrix}$$

$$\begin{pmatrix} q_1^2(a+b+c) + a(q_2^2 + q_3^2) - \rho\omega^2 & bq_1 q_2 & bq_1 q_3 \\ bq_2 q_1 & q_2^2(a+b+c) + a(q_1^2 + q_3^2) - \rho\omega^2 & bq_2 q_3 \\ bq_3 q_1 & bq_3 q_2 & q_3^2(a+b+c) + a(q_1^2 + q_2^2) - \rho\omega^2 \end{pmatrix}$$

$$a + b + c = c_{11}$$

$$\begin{pmatrix} c_{11} q_1^2 + c_{44}(q_2^2 + q_3^2) - \rho\omega^2 & (c_{12} + c_{44}) q_1 q_2 & (c_{12} + c_{44}) q_1 q_3 \\ (c_{12} + c_{44}) q_2 q_1 & c_{11} q_2^2 + c_{44}(q_1^2 + q_3^2) - \rho\omega^2 & (c_{12} + c_{44}) q_2 q_3 \\ (c_{12} + c_{44}) q_3 q_1 & (c_{12} + c_{44}) q_3 q_2 & c_{11} q_3^2 + c_{44}(q_1^2 + q_2^2) - \rho\omega^2 \end{pmatrix} =: (M)$$

$$\omega \text{ bestimmen aus}: \ |\mathrm{M}| = 0$$

$$q \parallel [100] \Rightarrow q_1 \neq 0,\ q_2 = q_3 = 0$$

$$\begin{vmatrix} c_{11} q_1^2 - \rho\omega^2 & 0 & 0 \\ 0 & c_{44} q_1^2 - \rho\omega^2 & 0 \\ 0 & 0 & c_{44} q_1^2 - \rho\omega^2 \end{vmatrix} = \left(c_{11} q_1^2 - \rho\omega^2\right)\left(c_{44} q_1^2 - \rho\omega^2\right)^2 = 0$$

$$\Rightarrow\ \omega_1 = \sqrt{\frac{c_{11}}{\rho}} q_1,\ \omega_{2,3} = \sqrt{\frac{c_{44}}{\rho}} q_1$$

Es muss gelten : $(\mathrm{M}) \cdot \vec{\mathrm{e}} = 0$

$\omega_1 \Rightarrow \vec{e}_1 = (1, 0, 0) \Rightarrow$ longitudinale Welle

$\omega_2 \Rightarrow \vec{e}_2 = (0, 1, 0) \Rightarrow$ transversale Welle

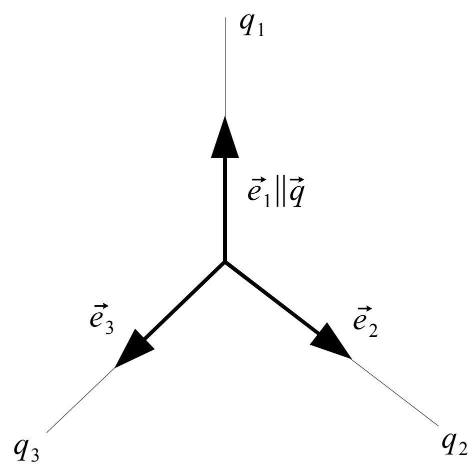
$\omega_3 \Rightarrow \vec{e}_3 = (0, 0, 1) \Rightarrow$ transversale Welle

Phasengeschwindigkeiten :

$$v = \frac{\omega}{|\vec{q}|}$$

$$v_1 = \frac{\omega_1}{q_1} = \sqrt{\frac{c_{11}}{\rho}},\ \mathrm{Al}: \sqrt{\frac{1,143 \cdot 10^{11}\mathrm{kg/m/s}^2}{2698\mathrm{kg/m}^3}} = 6,51\mathrm{km/s},\ \mathrm{Si}: \sqrt{\frac{1,66 \cdot 10^{11}\mathrm{kg/m/s}^2}{2326,3\mathrm{kg/m}^3}} = 8,45\mathrm{km/s}$$

$$v_{2,3} = \frac{\omega_{2,3}}{q_1} = \sqrt{\frac{c_{44}}{\rho}},\ \mathrm{Al}: \sqrt{\frac{0,316 \cdot 10^{11}\mathrm{kg/m/s}^2}{2698\mathrm{kg/m}^3}} = 3,69\mathrm{km/s},\ \mathrm{Si}: \sqrt{\frac{0,796 \cdot 10^{11}\mathrm{kg/m/s}^2}{2326,3\mathrm{kg/m}^3}} = 5,85\mathrm{km/s}$$



$$q \parallel [110] \Rightarrow q_1 = q_2 \neq 0,\ q_3 = 0$$

$$\begin{vmatrix} (c_{11}+c_{44})q_1^2 - \rho\omega^2 & (c_{12}+c_{44})q_1^2 & 0 \\ (c_{12}+c_{44})q_1^2 & (c_{11}+c_{44})q_1^2 - \rho\omega^2 & 0 \\ 0 & 0 & 2c_{44}q_1^2 - \rho\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow\ \omega_1 = \sqrt{\frac{2c_{44}}{\rho}}q_1,\ \omega_2 = \sqrt{\frac{c_{11}-c_{12}}{\rho}}q_1,\ \omega_3 = \sqrt{\frac{2c_{44}+c_{11}+c_{12}}{\rho}}q_1$$

Es muss gelten : $(\mathrm{M}) \cdot \vec{\mathrm{e}} = 0$

$\omega_1 \Rightarrow \vec{e}_1 = (0,0,1) \Rightarrow$ transversale Welle

$\omega_2 \Rightarrow \vec{e}_2 = \frac{1}{\sqrt{2}}(-1,1,0) \Rightarrow$ transversale Welle

$\omega_3 \Rightarrow \vec{e}_3 = \frac{1}{\sqrt{2}}(1,1,0) \Rightarrow$ longitudinale Welle

Phasengeschwindigkeiten :

$$v_1 = \frac{\omega_1}{\sqrt{2} \cdot q_1} = \sqrt{\frac{c_{44}}{\rho}},$$

$$\text{Al}: \sqrt{\frac{0,316 \cdot 10^{11}\text{kg/m/s}^2}{2698\text{kg/m}^3}} = 3,42\text{km/s},\ \text{Si}: \sqrt{\frac{0,796 \cdot 10^{11}\text{kg/m/s}^2}{2326,3\text{kg/m}^3}} = 5,85\text{km/s}$$

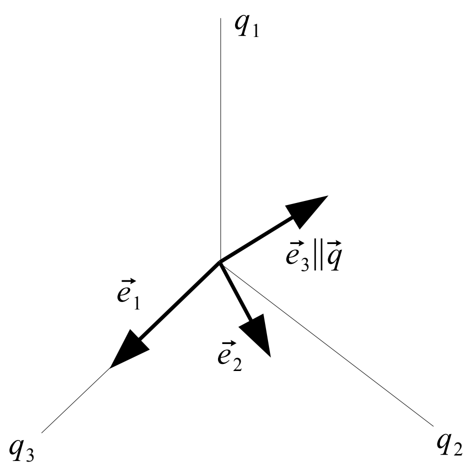
$$v_2 = \frac{\omega_2}{\sqrt{2} \cdot q_1} = \sqrt{\frac{c_{11} - c_{12}}{2 \cdot \rho}},$$

$$\text{Al}: \sqrt{\frac{(1,143 - 0,619) \cdot 10^{11}\text{kg/m/s}^2}{2 \cdot 2698\text{kg/m}^3}} = 3,12\text{km/s},\ \text{Si}: \sqrt{\frac{(1,66 - 0,639) \cdot 10^{11}\text{kg/m/s}^2}{2 \cdot 2326,3\text{kg/m}^3}} = 4,68\text{km/s}$$

$$v_3 = \frac{\omega_3}{\sqrt{2} \cdot q_1} = \sqrt{\frac{2c_{44} + c_{11} + c_{12}}{2 \cdot \rho}},$$

$$\text{Al}: \sqrt{\frac{(2 \cdot 0,316 + 1,143 + 0,619) \cdot 10^{11}\text{kg/m/s}^2}{2 \cdot 2698\text{kg/m}^3}} = 6,66\text{km/s}$$

$$\text{Si}: \sqrt{\frac{(2 \cdot 0,796 + 1,66 + 0,639) \cdot 10^{11}\text{kg/m/s}^2}{2 \cdot 2326,3\text{kg/m}^3}} = 9,14\text{km/s}$$



$$q \parallel [111] \Rightarrow q_1 = q_2 = q_3 \neq 0$$

$$\begin{vmatrix} (2c_{44} + c_{11})q_1^2 - \rho\omega^2 & (c_{12} + c_{44})q_1^2 & (c_{12} + c_{44})q_1^2 \\ (c_{12} + c_{44})q_1^2 & (2c_{44} + c_{11})q_1^2 - \rho\omega^2 & (c_{12} + c_{44})q_1^2 \\ (c_{12} + c_{44})q_1^2 & (c_{12} + c_{44})q_1^2 & (2c_{44} + c_{11})q_1^2 - \rho\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow\ \omega_1 = \sqrt{\frac{4c_{44} + c_{11} + 2c_{12}}{\rho}} q_1,\ \omega_{2,3} = \sqrt{\frac{c_{44} - c_{12} + c_{11}}{\rho}} q_1$$

Es muss gelten : $(\text{M}) \cdot \vec{e} = 0$

$$\omega_1 \Rightarrow \vec{e}_1 = \frac{1}{\sqrt{3}}(1,1,1) \Rightarrow \text{ longitudinale Welle}$$

$$\omega_2 \Rightarrow \vec{e}_2 = \frac{1}{\sqrt{2}}(-1,0,1) \Rightarrow \text{ transversale Welle}$$

$\omega_3 \Rightarrow \vec{e}_3 = \frac{1}{\sqrt{2}}(-1,1,0) \Rightarrow$ transversale Welle

Phasengeschwindigkeiten :

$$v_1 = \frac{\omega_1}{\sqrt{3}\cdot q_1} = \sqrt{\frac{4c_{44}+c_{11}+2c_{12}}{3\cdot\rho}},$$

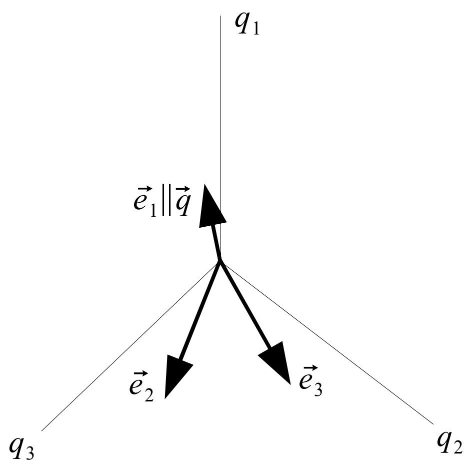
$$\text{Al}: \sqrt{\frac{(4\cdot 0,316+1,143+2\cdot 0,619)\cdot 10^{11}\text{kg/m/s}^2}{3\cdot 2698\text{kg/m}^3}} = 6,71\text{km/s}$$

$$\text{Si}: \sqrt{\frac{(4\cdot 0,796+1,66+2\cdot 0,639)\cdot 10^{11}\text{kg/m/s}^2}{3\cdot 2326,3\text{kg/m}^3}} = 9,37\text{km/s}$$

$$v_{2,3} = \frac{\omega_{2,3}}{\sqrt{3}\cdot q_1} = \sqrt{\frac{c_{44}-c_{12}+c_{11}}{3\cdot\rho}},$$

$$\text{Al}: \sqrt{\frac{(0,316-0,619+1,143)\cdot 10^{11}\text{kg/m/s}^2}{3\cdot 2698\text{kg/m}^3}} = 3,22\text{km/s}$$

$$\text{Si}: \sqrt{\frac{(0,796-0,639+1,66)\cdot 10^{11}\text{kg/m/s}^2}{3\cdot 2326,3\text{kg/m}^3}} = 5,10\text{km/s}$$



Man sieht, dass sich die longitudinalen Wellen schneller als die transversalen ausbreiten.